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MATHEMATICAL MODELING OF COMBAT OPERATIONS WITH PLANNING OF TARGET DISTRIBUTION COEFFICIENTS FOR ONE OF THE OPPOSING SIDES

The subject *of the study is models of combat operations.* ***The purpose*** *of this study is to build mathematical and computer models to identify the optimal target distribution coefficients of the active side in order to inflict the greatest losses on the enemy at a given time for the case when the sides have several types of weapons and direct fire at each type of enemy combat units with some target distribution coefficients.* ***Objectives:*** *1) to consider the case when the active side has one type of weapon, and the other side has two dissimilar types of weapons; 2) to generalize the problem to the case when the active side has n dissimilar types of weapons, and the other side has m dissimilar types of weapons; 3) to conduct a detailed study of the solution of the problem in the case when the sides have two dissimilar types of weapons. The following* ***results*** *were obtained: 1) a mathematical and computer models for the first case of the problem were built, the admissibility of the problem parameters was analyzed, and the corresponding numerical calculations were presented; 2) a mathematical model for the second case of the problem was created, the admissibility of the problem parameters was analyzed; 3) a mathematical and computer model for the third case was built and studied, the corresponding numerical calculations were presented, and the analysis was performed.* ***Conclusions.*** *The examples considered in this paper illustrate that, using a computer model, it is possible to predict how to allocate combat resources to fight the enemy to achieve success in combat at a given time if the enemy has two types of combat units. The problem has a solution in the general case, which requires maximizing a function of many variables at admissible lattice nodes, whose coordinates are the first side's objective coefficients. The paper shows how to find these admissible nodes. Numerical calculations before generalizing the problem demonstrate that it is possible by changing the time to predict the battle even in cases where the parties have several types of weapons.*

Keywords: *optimization mathematical model of combat dynamics; combat resources; velocity of impact; target distribution coefficient; intact combat units; striking potential.*

1. Introduction

1.1. Motivation

Mathematical models of combat operations are important primarily as a means of predicting their outcome. For constructing such models it is important to take into account both the dynamics of combat operations and the ultimate goals of the opposing sides.

In recent years, there has been an evolution in approaches to warfare (frontline fragmentation, widespread use of unmanned systems, multi-domain interaction), which has been reflected in the modernization of NATO doctrines [1]. In this regard, our approach should be understood as a generalized optimization model that uses the aggregated parameter “velocity of impact” to reflect the combined impact of both quantitative and qualitative factors (classical rate of fire, ammunition accuracy, remote unmanned platforms, electronic warfare means, etc.). Further generalizations of the model (in particular, the introduction of spatial coefficients or the modeling of “gray zones”) will allow for a better reflection of current doctrinal transformations.

To fully describe the picture on the battlefield, it is necessary to take into account various factors that influence the outcome: the importance of targets, the presence of target protection, the activity of drones, the formation of asymmetry through the use of different types of weapons, and the optimal distribution of resources. The influence of each of these factors is the subject of a separate study. Taking their impact into account simultaneously is an extremely difficult and rather impossible task. Therefore, it is advisable to study the impact of each factor separately, naturally assuming that the impact of others is reflected in such an integral indicator as velocity of impact, which, for simplicity, should be considered constant at the first stage. Velocity of impact is understood as an aggregate indicator of a unit's strike effectiveness, which is the product of the following indicators: the unit's base rate of fire (number of shots per unit of time), probability of hit (responsible for accuracy of hit), range multiplier (allows missiles/UAVs to be detected), guidance multiplier (allows reconnaissance/guidance data to be maintained). However, in the rest of the text, instead of the term:

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aggregated strike effectiveness indicator, we will use the term: velocity of impact. At the same time, if we do not take into account missiles/UAVs and do not take into account reconnaissance/guidance data support (i.e. the corresponding multipliers will be equal to one), then the velocity of impact will coincide with the classic effective rate of fire of a combat unit.

The work is devoted to studying the impact of optimal resource allocation on the outcome of combat, taking into account asymmetry due to the use of different types of weapons by the sides. This distribution is determined by target distribution coefficient, and asymmetry is taken into account by setting the velocity of impact of combat units of all types of weapons of both sides for each type of enemy weapon. That is, for the optimal distribution of resources, when the enemy has several types of weapons, it is necessary to plan in which parts to direct your resources (and they can also be of several types) in battle with different types of enemy resources. To solve this problem, target distribution coefficients (shares) of combat units of each type of one side for each type of combat units of the other side are introduced, which are parameters of the objective function, and the function of many variables is investigated for the maximum as a function of enemy losses. The admissibility of target distribution coefficients is determined, and the corresponding optimal target distribution coefficients of the active side and the greatest enemy losses at a given time moment are found, which correspond to these optimal distribution shares of the active side's combat resources. In addition, the total striking potential of both sides is introduced, and its dependence on optimal target distribution coefficients is determined for the active side. It should be noted that the problems considered in this paper are of such a general nature that their solutions ensure an optimal allocation of resources both in cases of significant asymmetry on the part of the attacking side (where the attacking side’s total strike potential exceeds that of the enemy) and when the enemy holds the advantage (the total strike potential of the active side is less than the total strike potential of the enemy). In the latter case, taking into account the number of combat units of various subunits and the strike speed of all types of combat units against all types of enemy combat units, the solution to the problem also yields a result. Namely, in what proportions should the active side’s combat resources be optimally allocated to combat different types of enemy combat units so that the enemy suffers the greatest total losses.

1.2. State of the art

To model the dynamics of combat operations, various versions of the Lanchester equations are widely used. The work [2] is devoted to mathematical models of combat that help predict the course and outcome of military confrontation. The author explains how two classic Lanchester models are applied in military affairs: the quadratic law of direct fire and the linear law of indirect fire, and demonstrates them with historical examples: operation “Desert Storm” (when the US used massive artillery to destroy enemy batteries), the Battle of Iwo Jima (Japan, US), the Battle of Alam (Mexico, US), and the transformation of indirect fire into direct fire with the use of drones in Ukraine.

The relevance of Lanchester-type models for describing the dynamics of combat operations, despite their long history, remains high today, as evidenced, in particular, by the following publications. Article [3] presents a probabilistic model in three areas of military capabilities: defense, awareness, and engagement. The authors of works [4, 5] use Lanchester models to study the optimal allocation of combat resources. In [6], game theory issues (Nash equilibrium) are addressed. The study [7] considers a model of an offensive air-ground combat operation, using approximate methods of numerical integration based on the Lanchaster law. Subsequent works propose extensions of the classical Lanchester models: in [8], consideration of a third party to the conflict; in [9, 10], variants of multi-battle conflicts. The authors of [11, 12] propose models with the deployment of troops in space and cognitive modeling of combat operations.

The paper [13] presents a mathematical model and a method of computer simulation of local combat engagements, implemented in the SimCombCalculator program. A modified version of the Lanchester model is used to describe the course of local combat, as well as a two-level logistic model that reproduces the processes of supplying ammunition and material resources.

Study [14] developed a dynamic model for assessing air combat situations based on methods for extracting knowledge about the situation and optimizing weight coefficients. The article [15] considers an approach to decision-making in autonomous air combat that combines simulation of team interaction and the FRV-DDPG deep deterministic gradient policy control algorithm.

Article [16] proposes an improved method for forming a contingency plan (COA) based on the construction and analysis of a network of operational tasks. The authors present an integrated model that combines hierarchical task networks (HTN) and influence networks (IN).

In their works, for example [17, 18], the authors propose optimization problem formulations that take into account the goal of combat: to inflict the greatest losses on the enemy at a given moment in time. In [17], combat is considered in two areas of contact and is conducted in several stages, during which the combat resources of the active side are distributed at the beginning of the battle, then transferred from one area to another in the second stage of the battle, and in the third stage of the battle, the available reserves of the active side are distributed among the areas of contact in order to inflict the greatest losses on the enemy. The solution uses dynamic programming in combination with solving Lanchester's differential equations at a certain stage of the battle. The parameters of the objective functions are, respectively, the amount of resources that are distributed, transferred from one area to another, and from the reserve of the active side at a certain stage of the battle. In [18], a battle between two sides is considered, and the optimal distribution of combat units of one side's unit for fighting three different enemy units is found. The problem in this formulation is a special case of a more complex problem solved in this work, where any number of units of the opposing sides is assumed.

In this paper [19], a simulation model of combat operations is investigated in which one side has homogeneous combat units of a single type, while the other has two types of homogeneous combat units, with the guidance coefficients of the side (which possesses a single type of weaponry) being specified for the two types of combat units of the other side. The solution to the corresponding system of three differential equations is presented using the operational method. However, work [19], firstly, does not take into account the objective of the battle: to inflict the greatest possible losses on the enemy; and secondly, considers only a specific case of the balance of forces: one type of combat units against two types of combat units.

1.3. Objectives and tasks

The purpose of this work is to address the following problems:

1) to develop a mathematical model and, using it, to solve the problem of maximizing the losses of one of the two sides to a given time moment by optimizing the target distribution coefficients of the first side, provided that it has homogeneous combat units of one type and directs fire at the combat units of the other side, that has homogeneous combat units of two types; 2) to find out the admissibility of the target distribution coefficients of the active side; 3) to build a mathematical model of combat in general and to explain its solution by example; 4) to build computer models of the considered tasks using Python programming, for which purpose to create appropriate solution algorithms; 5) to present the results of the corresponding numerical calculations.

Solving these problems significantly complements and generalizes the results obtained in [18] and [19]. Specifically, it takes into account the objective of the battle and generalizes a specific case of combat. This, in turn, makes it possible to make decisions in combat conditions using a more sophisticated tool.

Problems 1) and 2) are solved in Section 2 of the article. Problem 3) is solved in Sections 4 and 5, namely, in Section 4 a mathematical model of combat is formulated for the case when the active side has n dissimilar types of weapons, and the other side has m dissimilar types of weapons; the admissibility of the parameters of the problem in the general case and as an example in the case of $n = 1, m = 3$, is described, and Section 5 presents the analysis and solution of the problem for the case of $n = 2, m = 2$. Problem 4) of the article is presented in Sections 3 and 5 with algorithms for solving problem A and the problem for the case $n = 2, m = 2$ in the form of flowcharts, on the basis of which programs in the Python programming language were created. Finally, the realization of problem 5) is the corresponding numerical calculations in the form of an example and a table of results for task A (Section 3) and a table of results for the case $n = 2, m = 2$ (Section 5).

2. The theoretical results. Problem A

Problem A statement:

1. The battle between two groups - sides 1 and 2 - is considered. Side 1 has homogeneous combat units of one type in the amount of N_1, side 2 has homogeneous combat units of two types in the amount of N_2, N_3, respectively. Units of different types are heterogeneous.
2. The average velocity of impact of a combat unit of side 1 for combat units of side 2 of the first type α_{11} rounds per unit of time, for combat units of side 2 of the second type α_{12} rounds per unit of time. The average velocity of impact of a combat unit of side 2 of the first type β_{11} rounds per unit of time, of the second type β_{21} rounds per unit of time.
3. The battle lasts until the time t_1 (if any type of combat resource is completely destroyed, the battle is considered over).
4. Lanchester's systems of differential equations are used to model the dynamics of combat. It is believed that combat takes place in a "highly organized battle".

We need to find out in what proportions the first side needs to allocate its combat resources to fight the two types of combat resources of side 2 to achieve success, i.e., so that at the time t_1 side 2 has the greatest losses.

To solve this problem, let us denote by K the proportion of combat units of side 1 that must be sent to fight the first type combat units of side 2 to achieve success (we call this parameter the first target distribution coefficient of side 1). Then $1-K$ is the proportion of combat units of side 1 that must be sent to fight the second type combat units of side 2 to achieve success (we call this parameter the second target distribution coefficient of side 1). We have that $0 < K < 1$. It is necessary to maximize the loss function of side 2 at time t_1 by the parameter K. In this case, the number of undamaged combat units of both sides as a function of time is found as the corresponding solutions of the Lanchester differential equations with the appropriate initial conditions.

To draw up a system of differential equations, let $y_1(t)$ (respectively, $y_2(t)$ and $y_3(t)$) be the average of intact combat units of side 1 (side 2 of the first and, respectively, the second types) at time t. The scheme of the battle is shown in Fig. 1.

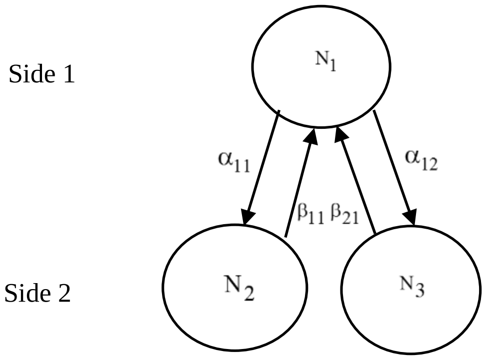


Fig. 1. Scheme of the battle between moments of time $t_0 = 0$ and $t_1 > t_0$.

According to the conditions of "highly organized combat" [20] and taking into account the first and second target distribution coefficients of side 1, the system of Lanchester's differential equations with respect to the average of groups (remaining intact at time t), given the initial conditions, is as follows:

$$\begin{cases} y_1' = -\beta_{11}y_2 - \beta_{21}y_3; \\ y_2' = -K\alpha_{11}y_1; \\ y_3' = -(1-K)\alpha_{12}y_1; \end{cases}$$

$$y_1(0) = N_1;$$
$$y_2(0) = N_2;$$
$$y_3(0) = N_3.$$

Let us define

$$a(K) = K\alpha_{11}\beta_{11} + \alpha_{12}\beta_{21}(1-K),$$
$$b = \beta_{21}N_3 + N_2\beta_{11}.$$

Solving the Cauchy problem above, we get that at the moment t_1 the following number of combat units of sides 1,2 remain intact:

$$y_1(t_1, K) = \frac{N_1\sqrt{a(K)} - b}{2\sqrt{a(K)}} \cdot e^{t_1\sqrt{a(K)}} + \\ + \frac{N_1\sqrt{a(K)} + b}{2\sqrt{a(K)}} \cdot e^{-t_1\sqrt{a(K)}}, \qquad (1)$$

$$y_2(t_1, K) = \frac{N_2 a(K) - K\alpha_{11}b}{a(K)} + \\ + \frac{K\alpha_{11}\left(b - N_1\sqrt{a(K)}\right)}{2a(K)} e^{t_1\sqrt{a(K)}} + \\ + \frac{K\alpha_{11}\left(b + N_1\sqrt{a(K)}\right)}{2a(K)} e^{-t_1\sqrt{a(K)}}, \qquad (2)$$

$$y_3(t_1, K) = \frac{N_3 a(K) - (1-K)\alpha_{12}b}{a(K)} + \\ + \frac{(1-K)\alpha_{12}\left(b - N_1\sqrt{a(K)}\right)}{2a(K)} \cdot e^{t_1\sqrt{a(K)}} + \\ + \frac{(1-K)\alpha_{12}\left(b + N_1\sqrt{a(K)}\right)}{2a(K)} \cdot e^{-t_1\sqrt{a(K)}}. \qquad (3)$$

Now we need to find the value of the parameter K so that the losses of side 2 at time t_1 are the largest. The losses of side 2 from time t_0 to time t_1 are denoted by $w(K)$, then

$$w(K) = (N_2 - y_2(t_1, K)) + (N_3 - y_3(t_1, K)). \qquad (4)$$

Accordingly, the largest losses of the side by the time of t_1:

$$W = \max_{K \in U_{n_1,n_2} \subset (0;1)} w(K). \qquad (5)$$

It is found that the maximum of the function $w(K)$ is necessary and sufficient to consider on a finite set of points U_{n_1,n_2}, belonging to the interval $(0;1)$, based on the following:

1) the number of combat units of side 1 sent to fight with each type of combat units of side 2 must be a positive integer;

2) the number of undamaged units of side 2, calculated by formulas (2), (3), must be non-negative;

3) side 1 must have at least one undamaged unit at time t_1, because this side is active. The set of points U_{n_1,n_2}, for which the maximum in (5) is to be found, is defined as follows: $U_{n_1,n_2} = \left\{\frac{i}{N_1}\right\}_{i=n_1}^{i=n_2}$, where

$1 \le n_1 \le n_2 \le \frac{N_1 - 1}{N_1}, \quad n_1, n_2 \in \mathbb{N}$.

Let us consider the striking combat potential of both sides accordingly:

$$P_1 = N_1 K \alpha_{11} + N_1 (1 - K) \alpha_{12},$$
$$P_2 = N_2 \beta_{11} + N_3 \beta_{21}.$$

Note that P_1 depends on parameter K, i.e., the optimal distribution coefficient K will correspond to the optimal striking potential at given values of the number of combat units of side 1 at the beginning of the battle N_1 and velocity of impact α_{11}, α_{12} in order to inflict the greatest losses on the enemy at time t_1.

3. Solving problem A

To solve problem A numerically, algorithm has been developed (Fig. 2) and a Python program.

Let us briefly explain how the algorithm works. The initial data of the problem are entered. Next, a loop is organized according to the parameter K in the range from the number $\frac{1}{N_1}$ to the number $\frac{N_1 - 1}{N_1}$ with a step $\frac{1}{N_1}$, in which the number of intact combat units of both

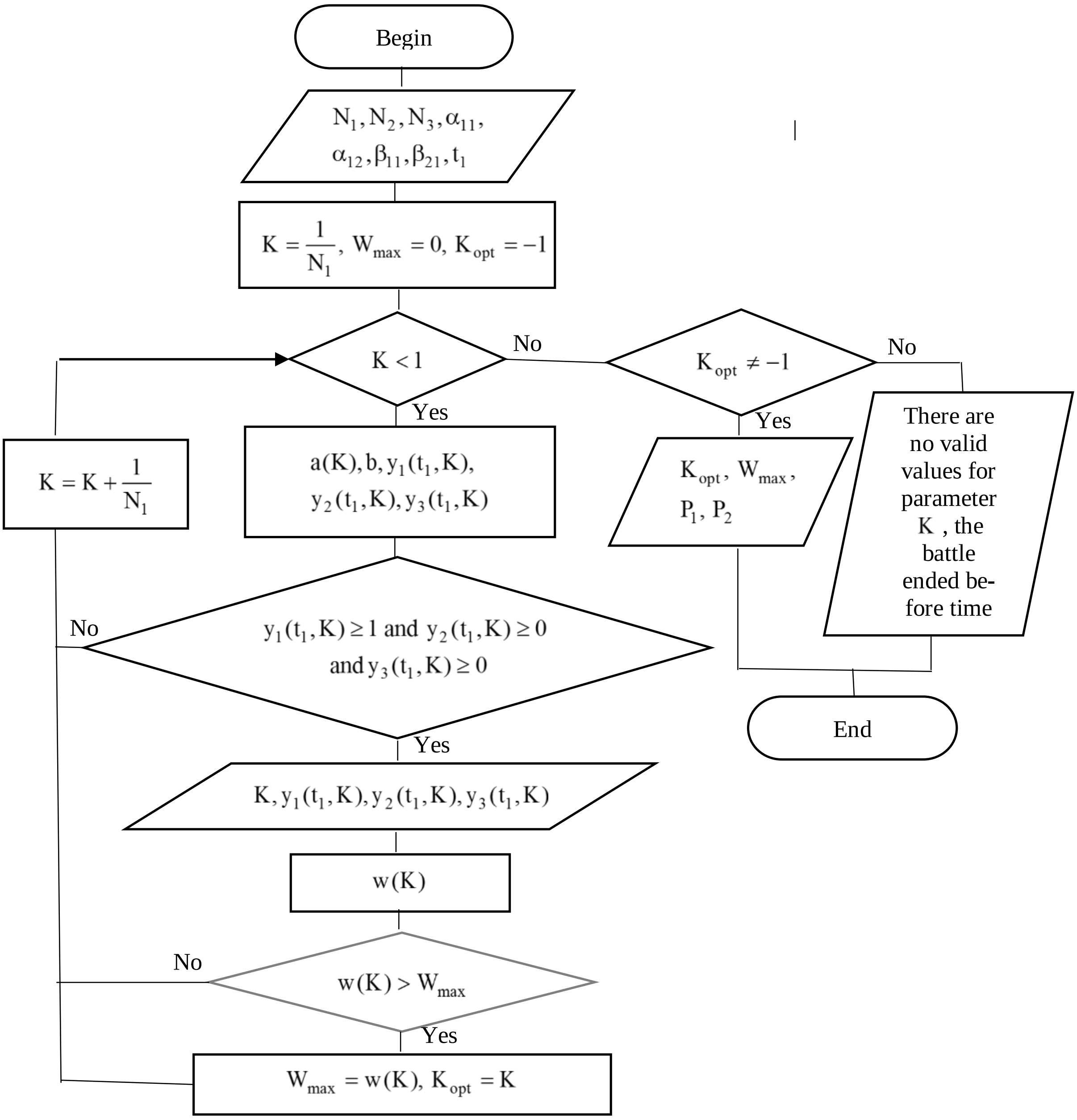


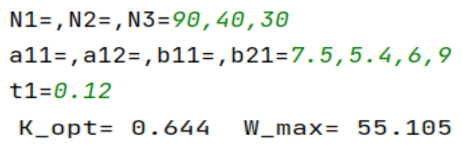
Fig. 2. Algorithm for solving the problem A

sides at the time t_1 is calculated. Next in the loop are the permissible values of the parameter K. For the permissible values of the parameter K, K and the corresponding numbers of intact combat units of both sides are printed. Moreover, the losses of side 2 are calculated and the largest losses of side 2 and the corresponding optimal value of the parameter are found. The peculiarity is that before the cycle the "indicator" $K_{opt} = -1$ is entered, if it does not change after the cycle by K, it means that there are no valid values of the parameter, i.e. the battle ended before the time t_1 (the time should be reduced). Otherwise, K_{opt}, P_1, P_2 is printed and the largest losses of side 2 are W_{max}.

Example. Let the battle take place under conditions 1-4. At the beginning of the battle, side 1 has 90 combat units, the average velocity of impact on side 2 combat units of the first type 7.5 shots per hour (the average of enemy combat units hit per unit of time by a combat unit of side 1), the second type 5.4 shots per hour. Side 2 has 40 combat units of the first type, their average velocity of impact 6 shots per hour; 30 combat units of the second type, their average velocity of impact 9 shots per hour. What proportions of the combat units of side 1 should be sent to fight with the combat units of side 2 of the first and second types, respectively, so the losses of side 2 at the time of 0.12 hours would be the greatest

The result is that the first optimal target distribution coefficient of side 1 is 0.644, and the second is 0.356. This means that to achieve the success of the first side, i.e., for side 2 to have the greatest losses at time $t = 0.12$ h, 58 combat units of side 1 should be sent to fight with the first type combat units of side 2 and 32 combat units of side 1 should be sent to fight with the second type combat units of side 2. The largest losses of side 2 at the time of 0.12 hours are 55 combat units. The optimal value of the target distribution coefficient 0.644 corresponds to the striking potential of side 1 $P_1 = 608$, while the striking potential of side 2 is $P_2 = 510$, where $P_1 > P_2$.

Results of solving Example 1 (Fig. 3).



```
N1=,N2=,N3=90,40,30
a11=,a12=,b11=,b21=7.5,5.4,6,9
t1=0.12
 K_opt= 0.644  W_max= 55.105
```

Fig. 3. Results of solving Example 1

Here are the results of solving problem A for several initial data (Table 1).

It is concluded that the target distribution coefficients cannot be specified in advance and must be calculated.

4. Problem B. Generalization of problem A

Let the battle take place under the following conditions:

1. Side 1 has n types of combat resources (i-th type has N_i $(i = 1, 2, ..., n)$ combat units), and the other side has m types of combat resources (j-th type has N_{n+j} $(j = 1, 2, ..., m)$ combat units), and each type of combat unit of both sides fires at all types of combat units of the other side ($N_i \geq m$, $i = 1, 2, ..., n$; $N_{n+j} \geq n$, $j = 1, 2, ..., m$).

Table 1

Numerical calculations for solving problem A

№	N_1	N_2	N_3	α_{11}	α_{12}	β_{11}	β_{21}	t_1	K_{opt}	W_{max}	P_1	P_2
1.	90	50	30	7.5	9	12	8	0.13	0.5	55	742	840
2.	60	40	30	12	8.4	6	9	0.16	0.617	56	637	510
3.	80	30	20	6	8	10	6	0.1	0.625	44	540	420
4.	60	30	50	12	9	7.8	10.2	0.1	0.833	34	690	744
5.	60	50	30	12	9	7.8	10.2	0.1	0.983	39	717	696
6.	100	40	50	6	12	15	7.8	0.15	0.45	70	930	990
7.	100	40	50	6	12	15	13.5	0.12	0.27	59	1038	1275
8.	60	20	10	15	12	5	6	0.035	0.65	28	837	160
9.	50	15	10	14	16	8	7	0.032	0.6	22	740	190

2. Side 2 sends its combat resources with target distribution coefficients γ_{ji} (the share of combat resources of the j-th type of side 2 in the combat resources of the i-th type of side 1), the corresponding velocity of impact β_{ji} $j = 1, 2, ..., m,\ i = 1, 2, ..., n.$ At $n = 1$, the target distribution coefficients $\gamma_{j1} = 1$, $j = 1, 2, ..., m$.

3. Side 1 has the following velocity of impact α_{ij}, $i = 1, 2, ..., n,\ j = 1, 2, ..., m$ (i corresponds to the type of combat resources of side 1, j to the type of combat resources of side 2).

4. The battle takes place in a “highly organized battle” and lasts until the time t_1.

We need to find out what proportions of each type of combat resources of the first side should be sent to fight with all types of combat resources of the second side to achieve success, that is, so that at the time t_1 side 2 has the greatest losses.

Let us denote by K_{ij} the target distribution coefficient of the i-th type of combat resources of the first side by the j-th type of combat resources of the second side ($i = 1, 2, ..., n,\ j = 1, 2, ..., m$). We have that $0 < K_{ij} < 1$. The system of differential equations describing the battle, relative to the average of combat units of both sides that remained intact at the moment of time t, has the form (here $y_i\ (i = 1, 2, ..., n)$ is the average of combat units of the i-th type of the first side, $y_{n+j}\ (j = 1, 2, ..., m)$ is the average of combat units of the j-th type of the second side):

$$\begin{cases} y_1' = -\sum\limits_{j=1}^{m} \beta_{j1}\gamma_{j1} y_{n+j} \\ \dots\dots\dots\dots\dots\dots \\ y_n' = -\sum\limits_{j=1}^{m} \beta_{jn}\gamma_{jn} y_{n+j} \\ y_{n+1}' = -\sum\limits_{i=1}^{n} K_{i1}\alpha_{i1} y_i \\ \dots\dots\dots\dots\dots\dots \\ y_{n+m-1}' = -\sum\limits_{i=1}^{n} K_{i\,m-1}\alpha_{i\,m-1} y_i \\ y_{n+m}' = -\sum\limits_{i=1}^{n} \left(1 - \sum\limits_{l=1}^{m-1} K_{il}\right)\alpha_{im} y_i, \end{cases} \quad (6)$$

where

$$0 < K_{ij} < 1,\ i = 1, 2, ..., n,\ j = 1, 2, ..., m-1,$$

$$\sum_{l=1}^{m-1} K_{il} < 1,\ \ i = 1, ..., n\ .$$

Initial conditions:

$$y_i(0) = N_i\ \ (i = 1, 2, ..., n),$$
$$y_{n+j}(0) = N_{n+j}\ \ (j = 1, 2, ..., m). \quad (7)$$

To solve the problem we need to find the largest value of the second side loss function $(m-1)n$ of the variables

$$w(K_{11}, ..., K_{1m-1}, ..., K_{n1}, ..., K_{nm-1}) = \sum_{j=1}^{m} (N_{n+j} -$$
$$- y_{n+j}(t_1, K_{11}, ..., K_{1m-1}, ..., K_{n1}, ..., K_{nm-1}))$$

in the nodes of the "lattice" belonging to the interior of the $(m-1)n$-dimensional cube described by inequalities:

$$0 < K_{ij} < 1,\ \ i = 1, 2, ..., n,\ \ j = 1, 2, ..., m-1$$

and adding conditions

$$\sum_{l=1}^{m-1} K_{il} < 1,\ \ i = 1, ..., n\ .$$

In this case, on the K_{ij} axis, the corresponding coordinate of the node will take on the values $\frac{1}{N_i}, \frac{2}{N_i}, ..., \frac{N_i - m + 1}{N_i}$ (the last fraction is exactly that, since side 1 must send at least one unit of each type to fight with the combat resources of each type of side 2). But you need to take into account that: 1) all solutions of the Cauchy problem (6) with initial conditions (7), whose values count the number of undamaged combat units of side 2 at the moment of time t_1, must be non-negative; 2) side 1 must have at least one undamaged combat unit of each type, because it is the active side. Therefore, from the previously identified nodes, we must exclude the nodes for which the above conditions are not met. In this way, the valid nodes will be found, the coordinates of which are respectively valid values of the target distribution coefficients.

Note that if there are no valid nodes, then the problem is unsolvable for the data under consideration.

For example, consider the case of $n = 1,\ m = 3$ [18]. Then we have the following conditions: $0 < K_{11} < 1$, $0 < K_{12} < 1,\ K_{11} + K_{12} < 1$, with the corresponding coordinate of the node on the K_{11} axis and on the K_{12}

axis taking the values $\frac{1}{N_1}, \frac{2}{N_1}, ..., \frac{N_1-2}{N_1}$ (Fig. 4).

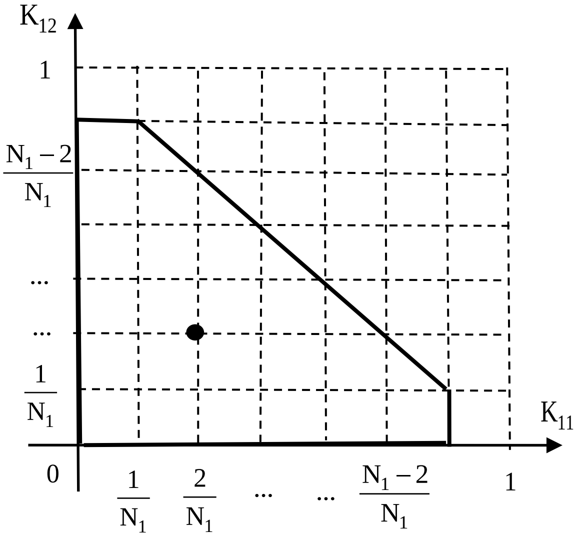


Fig. 4. "Lattice" for the case $n = 1, m = 3$.

In Fig. 4, the node is marked with a point.

That is, out of necessity, we first consider the nodes inside the polygon in Figure 4, and then it is enough to remove those nodes for which the inequalities

$$y_1(t_1, K_{11}, K_{12}) \geq 1,\ y_2(t_1, K_{11}, K_{12}) \geq 0,$$
$$y_3(t_1, K_{11}, K_{12}) \geq 0,\ y_4(t_1, K_{11}, K_{12}) \geq 0$$

are not satisfied to find valid nodes.

In the fourth section of the article, we will depict the corresponding "lattice" with nodes for the case of $n = 2, m = 2$ (Figure 6).

Let us consider the total striking potentials of both sides in problem B:

$$P_1 = \sum_{i=1}^{n} \sum_{j=1}^{m} N_i K_{ij} \alpha_{ij},$$

where

$$K_{im} = 1 - \sum_{l=1}^{m-1} K_{il}\ \ (i = 1, 2, ..., n);$$

$$P_2 = \sum_{j=1}^{m} \sum_{i=1}^{n} N_{n+j} \gamma_{ji} \beta_{ji}.$$

Note that the total striking potential of the first side, P_1, is the number of combat units that side 1 can strike at the beginning of the battle per unit of time if side 2 does not resist. Similarly, the total striking potential of the second side P_2 is the number of combat units that side 2 can strike at the beginning of the battle per unit of time if side 1 does not resist.

5. Solving problem B. Example

Consider problem B for the case $n = 2, m = 2$. Then the battle diagram will look like in Fig. 5.

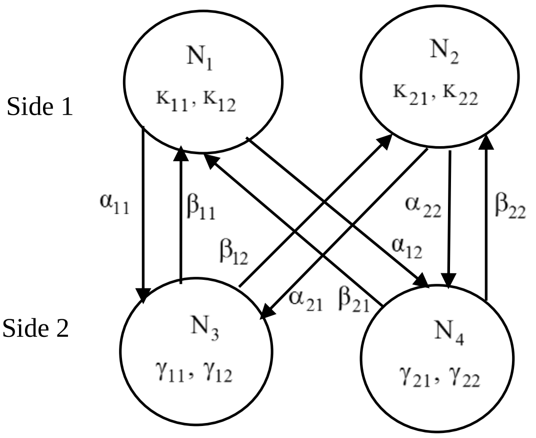


Fig. 5. Scheme of the battle for the case $n = 2, m = 2$

The corresponding system of differential equations, given the initial conditions, is as follows:

$$\begin{cases} y_1' = -\beta_{11}\gamma_{11}y_3 - \beta_{21}\gamma_{21}y_4; \\ y_2' = -\beta_{12}\gamma_{12}y_3 - \beta_{22}\gamma_{22}y_4; \\ y_3' = -K_{11}\alpha_{11}y_1 - K_{21}\alpha_{21}y_2; \\ y_4' = -(1-K_{11})\alpha_{12}y_1 - (1-K_{21})\alpha_{22}y_2; \end{cases} \tag{8}$$

$$\begin{matrix} y_1(0) = N_1;\ \ y_2(0) = N_2; \\ y_3(0) = N_3;\ \ y_4(0) = N_4; \end{matrix} \tag{9}$$

The Cauchy problem (8), (9) has a unique solution if and only if two conditions are met:

$$S = (1-K_{11})\alpha_{12}K_{21}\alpha_{21} - (1-K_{21})\alpha_{22}K_{11}\alpha_{11} \neq 0,$$

$$r = \beta_{11}\gamma_{11}\beta_{22}\gamma_{22} - \beta_{21}\gamma_{21}\beta_{12}\gamma_{12} \neq 0.$$

Let us define

$$a = (1-K_{11})\alpha_{12}\beta_{11}\gamma_{11} + (1-K_{21})\alpha_{22}\beta_{12}\gamma_{12},$$

$$b = (1-K_{11})\alpha_{12}\beta_{21}\gamma_{21} + (1-K_{21})\alpha_{22}\beta_{22}\gamma_{22},$$

$$c = K_{11}\alpha_{11}\beta_{11}\gamma_{11} + K_{21}\alpha_{21}\beta_{12}\gamma_{12},$$

$$d = K_{11}\alpha_{11}\beta_{21}\gamma_{21} + K_{21}\alpha_{21}\beta_{22}\gamma_{22},$$

$$\lambda_1 = \sqrt{\frac{b+c+\sqrt{(b-c)^2+4ad}}{2}},$$

$$\lambda_3 = \sqrt{\frac{\left|b+c-\sqrt{(b-c)^2+4ad}\right|}{2}}.$$

It should be noted that $a > 0,\ b > 0,\ c > 0,\ d > 0.$

Let us consider three cases:

1) $bc - ad > 0$, 2) $bc - ad < 0$, 3) $bc - ad = 0$.

Note that in the first case, the submodular expression under the root of λ_3 is positive, in the second case it is negative, and in the third case it is zero. It is obtained that in each of these cases, the Cauchy problem (8), (9) has its own solution.

Let us consider the first case. We denote

$$r_1 = \frac{\lambda_1^2 - b}{a}, \quad r_2 = \frac{\lambda_3^2 - b}{a},$$

$$g_1 = -K_{21}\alpha_{21} + (1-K_{21})\alpha_{22}r_1,$$
$$g_2 = -K_{21}\alpha_{21} + (1-K_{21})\alpha_{22}r_2,$$
$$g_3 = K_{11}\alpha_{11} - (1-K_{11})\alpha_{12}r_1,$$
$$g_4 = K_{11}\alpha_{11} - (1-K_{11})\alpha_{12}r_2.$$

Then at time t_1, the following number of combat units of sides 1,2 remain intact:

$$y_1(t_1, K_{11}, K_{21}) = \frac{\lambda_1}{S} g_1 \left(C_1 e^{\lambda_1 t_1} - C_2 e^{-\lambda_1 t_1}\right) + \frac{\lambda_3}{S} g_2 \left(C_3 e^{\lambda_3 t_1} - C_4 e^{-\lambda_3 t_1}\right),$$

$$y_2(t_1, K_{11}, K_{21}) = \frac{\lambda_1}{S} g_3 \left(C_1 e^{\lambda_1 t_1} - C_2 e^{-\lambda_1 t_1}\right) + \frac{\lambda_3}{S} g_4 \left(C_3 e^{\lambda_3 t_1} - C_4 e^{-\lambda_3 t_1}\right),$$

$$y_3(t_1, K_{11}, K_{21}) = r_1 \left(C_1 e^{\lambda_1 t_1} + C_2 e^{-\lambda_1 t_1}\right) + r_2 \left(C_3 e^{\lambda_3 t_1} + C_4 e^{-\lambda_3 t_1}\right),$$

$$y_4(t_1, K_{11}, K_{21}) = C_1 e^{\lambda_1 t_1} + C_2 e^{-\lambda_1 t_1} + C_3 e^{\lambda_3 t_1} + C_4 e^{-\lambda_3 t_1},$$

where

$$C_1 = \frac{1}{2}\left(\frac{aN_3 - ar_2N_4}{\lambda_1^2 - \lambda_3^2} + \frac{S}{\lambda_1}\frac{g_2N_2 - g_4N_1}{g_2g_3 - g_1g_4}\right),$$

$$C_2 = \frac{1}{2}\left(\frac{aN_3 - ar_2N_4}{\lambda_1^2 - \lambda_3^2} - \frac{S}{\lambda_1}\frac{g_2N_2 - g_4N_1}{g_2g_3 - g_1g_4}\right),$$

$$C_3 = \frac{1}{2}\left(\frac{aN_3 - ar_1N_4}{\lambda_3^2 - \lambda_1^2} + \frac{S}{\lambda_3}\frac{g_1N_2 - g_3N_1}{g_1g_4 - g_2g_3}\right),$$

$$C_4 = \frac{1}{2}\left(\frac{aN_3 - ar_1N_4}{\lambda_3^2 - \lambda_1^2} - \frac{S}{\lambda_3}\frac{g_1N_2 - g_3N_1}{g_1g_4 - g_2g_3}\right).$$

It should be noted that $g_1g_4 - g_2g_3 = \frac{S}{a}(\lambda_3^2 - \lambda_1^2) \neq 0$, вecause $S \neq 0,\ \lambda_1^2 \neq \lambda_3^2$.

Let us consider the second case. Let $r_1,\ g_1,\ g_3$ be defined as in the first case. In addition, we denote

$$r_3 = \frac{\lambda_3^2 + b}{a}, \quad g_5 = -K_{21}\alpha_{21} - (1-K_{21})\alpha_{22}r_3,$$
$$g_6 = K_{11}\alpha_{11} + (1-K_{11})\alpha_{12}r_3.$$

Then at time t_1, the following number of combat units of sides 1,2 remain intact:

$$y_1(t_1, K_{11}, K_{21}) = \frac{\lambda_1}{S} g_1 \left(C_1 e^{\lambda_1 t_1} - C_2 e^{-\lambda_1 t_1}\right) + \frac{\lambda_3}{S} g_5 \left(-C_3 \sin\lambda_3 t_1 + C_4 \cos\lambda_3 t_1\right),$$

$$y_2(t_1, K_{11}, K_{21}) = \frac{\lambda_1}{S} g_3 \left(C_1 e^{\lambda_1 t_1} - C_2 e^{-\lambda_1 t_1}\right) + \frac{\lambda_3}{S} g_6 \left(-C_3 \sin\lambda_3 t_1 + C_4 \cos\lambda_3 t_1\right),$$

$$y_3(t_1, K_{11}, K_{21}) = r_1 \left(C_1 e^{\lambda_1 t_1} + C_2 e^{-\lambda_1 t_1}\right) + r_3 \left(-C_3 \cos\lambda_3 t_1 - C_4 \sin\lambda_3 t_1\right),$$

$$y_4(t_1, K_{11}, K_{21}) = C_1 e^{\lambda_1 t} + C_2 e^{-\lambda_1 t} + C_3 \cos\lambda_3 t_1 + C_4 \sin\lambda_3 t_1,$$

where

$$C_1 = \frac{1}{2}\left(\frac{N_3 + r_3N_4}{r_1 + r_3} + \frac{S}{\lambda_1}\frac{g_5N_2 - g_6N_1}{g_3g_5 - g_1g_6}\right),$$

$$C_2 = \frac{1}{2}\left(\frac{N_3 + r_3 N_4}{r_1 + r_3} - \frac{S}{\lambda_1}\frac{g_5 N_2 - g_6 N_1}{g_3 g_5 - g_1 g_6}\right),$$

$$C_3 = \frac{r_1 N_4 - N_3}{r_1 + r_3}, \quad C_4 = \frac{S}{\lambda_3}\frac{g_1 N_2 - g_3 N_1}{g_1 g_6 - g_3 g_5}.$$

It should be noted that $g_3 g_5 - g_1 g_6 = S(r_1 + r_3) \neq 0$, because $S \neq 0,\ r_1 + r_3 = \frac{\lambda_1^2 + \lambda_3^2}{a} \neq 0$.

Let us consider the third case. Let r_1, g_1, g_3 be defined as in the first case. We also denote

$$r_4 = \frac{b}{a}, \quad g_7 = K_{21}\alpha_{21} + (1 - K_{21})\alpha_{22} r_4,$$

$$g_8 = K_{11}\alpha_{11} + (1 - K_{11})\alpha_{12} r_4.$$

Then at time t_1, the following number of combat units of sides 1,2 remain intact:

$$y_1(t_1, K_{11}, K_{21}) = \frac{\lambda_1}{S} g_1\left(C_1 e^{\lambda_1 t_1} - C_2 e^{-\lambda_1 t_1}\right) - \frac{g_7}{S} C_4,$$

$$y_2(t_1, K_{11}, K_{21}) = \frac{\lambda_1}{S} g_3\left(C_1 e^{\lambda_1 t_1} - C_2 e^{-\lambda_1 t_1}\right) + \frac{g_8}{S} C_4,$$

$$y_3(t_1, K_{11}, K_{21}) = r_1\left(C_1 e^{\lambda_1 t_1} + C_2 e^{-\lambda_1 t_1}\right) - r_4(C_3 + C_4 t_1),$$

$$y_4(t_1, K_{11}, K_{21}) = C_1 e^{\lambda_1 t_1} + C_2 e^{-\lambda_1 t_1} + C_3 + C_4 t_1,$$

where

$$C_1 = \frac{1}{2}\left(\frac{N_3 + N_4 r_4}{r_1 + r_4} + \frac{S}{\lambda_1}\frac{g_7 N_2 + g_8 N_1}{g_3 g_7 + g_1 g_8}\right),$$

$$C_2 = \frac{1}{2}\left(\frac{N_3 + N_4 r_4}{r_1 + r_4} - \frac{S}{\lambda_1}\frac{g_7 N_2 + g_8 N_1}{g_3 g_7 + g_1 g_8}\right),$$

$$C_3 = \frac{r_1 N_4 - N_3}{r_1 + r_4}, \quad C_4 = S\frac{g_1 N_2 - g_3 N_1}{g_1 g_8 + g_3 g_7}.$$

It should be noted that $g_3 g_7 + g_1 g_8 = -S(r_1 + r_4) \neq 0$, because $S \neq 0,\ r_1 + r_4 = \frac{\lambda_1^2}{a} \neq 0$.

To solve the problem, we need to find the largest value of the second side loss function of two variables

$$w(K_{11}, K_{21}) =$$

$$= N_3 + N_4 - y_3(t_1, K_{11}, K_{21}) - y_4(t_1, K_{11}, K_{21})$$

in the nodes of the “lattice” belonging to the interior of the square, while on the axis K_{ij} the corresponding coordinate of the node will take on the values, $\frac{1}{N_i}, \frac{2}{N_i}, ..., \frac{N_i - 1}{N_i}$, $i = 1, 2;\ j = 1$ (Fig. 6). In this case, the valid nodes are the following nodes, at the corresponding values of the coordinates K_{11}, K_{21}, which fulfill the conditions:

1) $S \neq 0$ and $r \neq 0$;

2) the number of undamaged combat units of side 2 of both types must be nonnegative;

3) side 1 must have at least one undamaged unit of each type at time t_1, because this side is active.

In Fig. 6, the lattice node with coordinates $\left(\frac{i}{N_1}; \frac{j}{N_2}\right)$ is represented by a point.

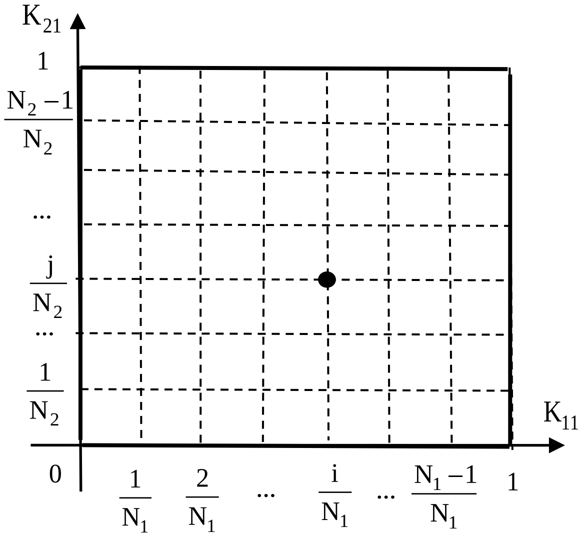


Fig. 6. "Lattice" for the case of $n = 2,\ m = 2$

The corresponding total striking potentials of both sides are calculated using the following formulas:

$$P_1 = N_1(K_{11}\alpha_{11} + K_{12}\alpha_{12}) + N_2(K_{21}\alpha_{21} + K_{22}\alpha_{22}),$$

where

$$K_{12} = 1 - K_{11},\ K_{22} = 1 - K_{21};$$

$$P_2 = N_3(\gamma_{11}\beta_{11} + \gamma_{12}\beta_{12}) + N_4(\gamma_{21}\beta_{21} + \gamma_{22}\beta_{22}).$$

To solve the problem numerically, we developed a Python program based on the algorithm (Fig. 7, 8).

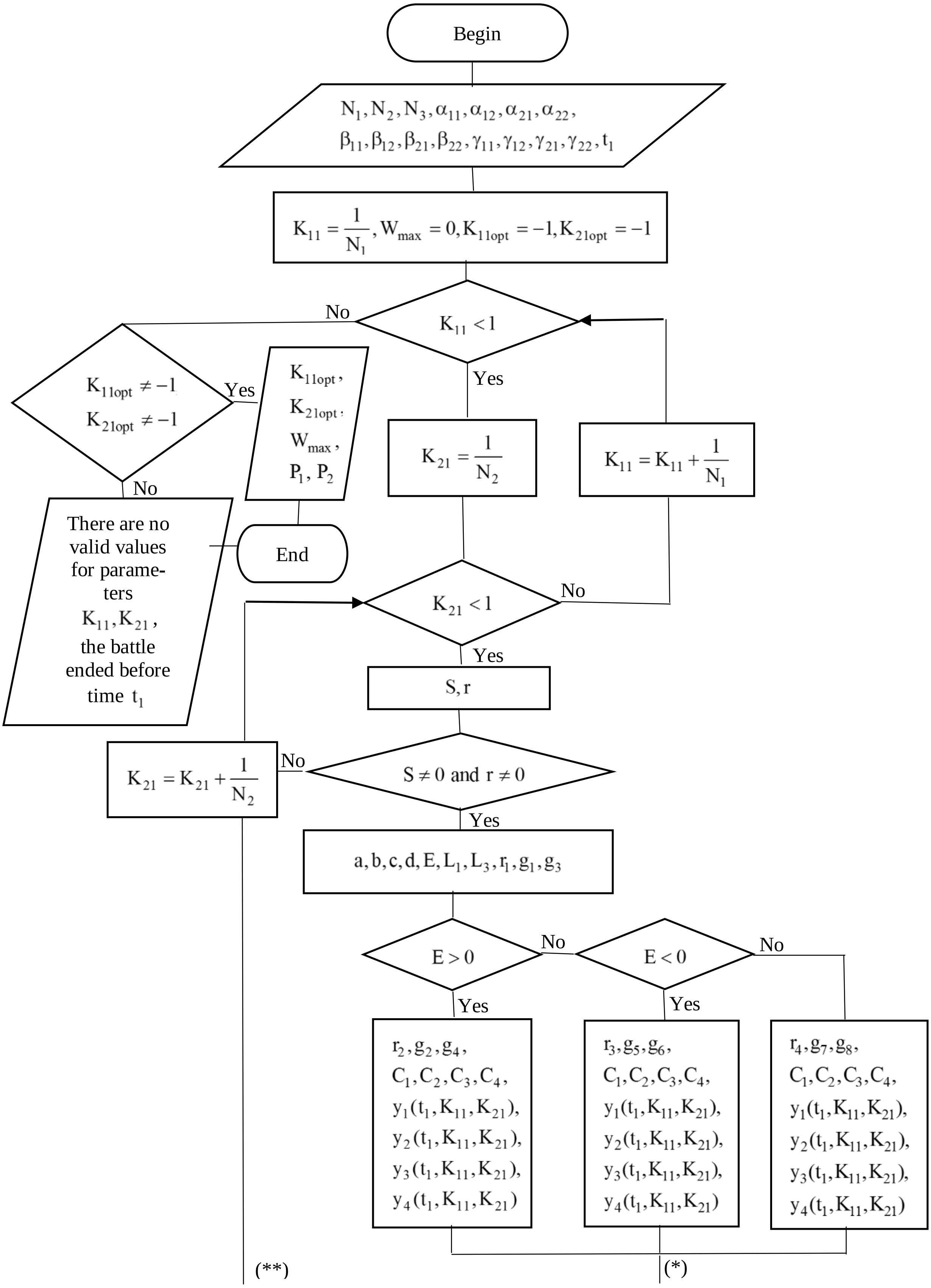


Fig. 7. Solution algorithm for the case $n = 2$, $m = 2$

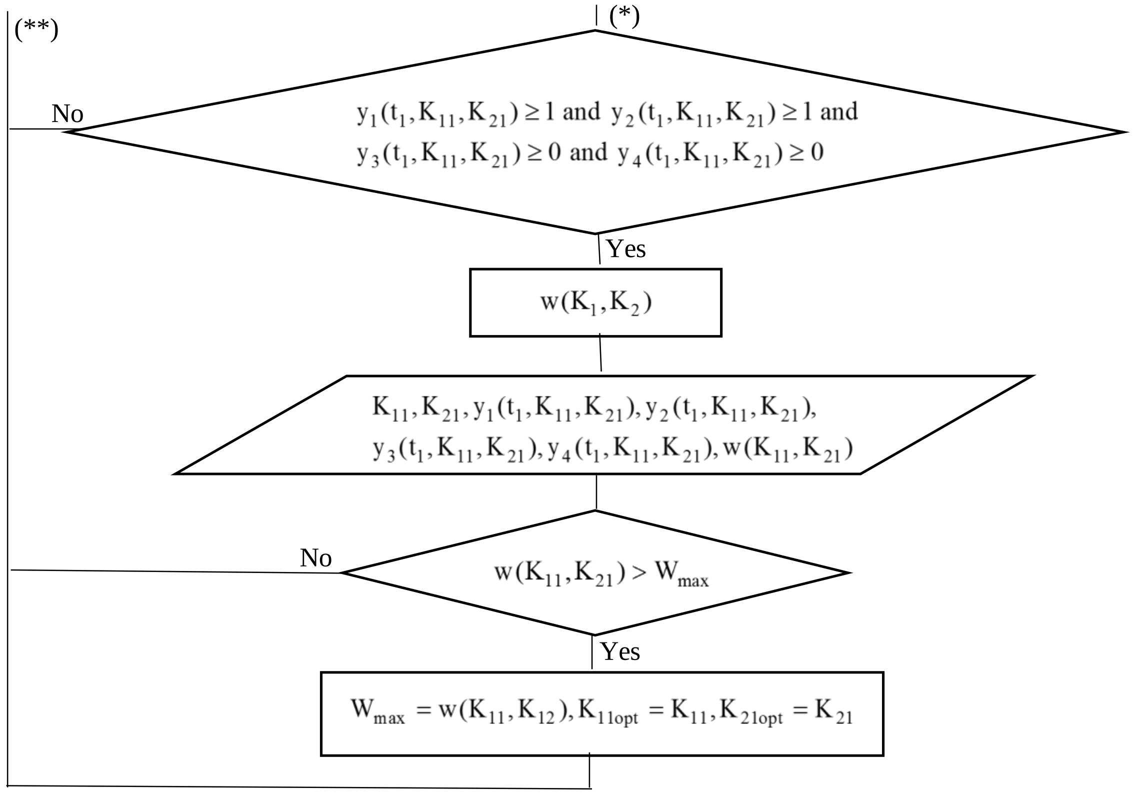


Fig. 8. End of the algorithm for solving the problem for the case $n = 2$, $m = 2$

Let us explain how the algorithm works. The data of the problem are entered and two nested cycles are organized by the parameter K_{11} in the range from the number $\frac{1}{N_1}$ to the number $\frac{N_1-1}{N_1}$ with a step $\frac{1}{N_1}$ and by the parameter K_{21} in the range from the number $\frac{1}{N_2}$ to the number $\frac{N_2-1}{N_2}$ with a step $\frac{1}{N_2}$, in which the values S and r are calculated. The condition of uniqueness of the system solution is checked. The values required to find the number of intact combat units of both sides at the time t_1 are calculated. Next comes the part of the algorithm where, depending on the three cases 1) $bc - ad > 0$, 2) $bc - ad < 0$, 3) $bc - ad = 0$, the number of intact combat units of both sides is calculated. It is formed by using the "if" and "else if" loops for cases 1)-3). Next, we find the permissible values of the target distribution coefficients K_{11} and K_{21}, print them, the number of undamaged units on both sides, and the corresponding enemy losses. Then we find the greatest enemy losses and the corresponding optimal K_{11} and K_{21}. At the beginning of the program, the "indicators" $K_{11opt} = -1$ and $K_{21opt} = -1$ are introduced; if they do not change after cycles through K_{11} and K_{21}, it means that there are no valid values for these parameters, that is, the battle ended before the time t_1 (the time should be reduced). Otherwise, K_{11opt}, K_{21opt}, P_1, P_2 and the largest losses of side 2 W_{max} are printed. Here are the results of numerical calculations for several initial data of the problem for the case $n = 2$, $m = 2$ (Table 2). The results of the calculations are given without data for certain types of weapons, but the calculations can be performed for them as well.

Thus, using a computer model, it is possible to find the optimal target distribution coefficients that will ensure maximum losses for one of the combatants. Moreover, by applying asymmetry in combat, characterized by the superiority of the striking potential of side 1 (for example, examples 3, 7, 11, 12), it is possible to distribute the combat resources of several types of side 1 among the combat resources of several types of side 2 in such a way that the losses of the enemy (side 2) will be the greatest. We note that the Python program provides optimal resource allocation even when the striking potential of side 1 is less than that of side 2 (e.g., example 6), i.e., it is possible in this case to inflict the greatest possible losses on the enemy.

Table 2

Numerical calculations of the solution for the case $n = 2,\ m = 2$

№	1	2	3	4	5	6	7	8	9	10	11	12
N_1	40	40	50	50	40	40	60	60	100	80	60	50
N_2	30	30	60	60	50	50	70	70	60	70	50	30
N_3	20	20	30	30	60	60	30	30	80	100	20	10
N_4	30	30	40	40	50	70	40	49	80	60	15	20
α_{11}	6	6	12	9	9,6	8.6	6	9	9	7.2	12	15
α_{12}	9	9	7.5	7.5	7.8	7.8	9	6.6	10.8	8.8	10	11
α_{21}	8.4	8.4	9	6.8	12	9	7.2	8.4	12	9	15	10
α_{22}	6	6	10.5	9.8	6	6	8.4	6	11	9.6	13	12.8
β_{11}	5.4	5.4	6	8.4	9	10	9	12	7.5	6.2	5	6
β_{12}	6	6	9	7.8	6	12	10.8	9	9	6.6	4	4
β_{21}	9	9	12	6	10.8	9.8	9.6	10.8	10.8	9.4	6	5
β_{22}	7.5	7.5	8.4	9	7.2	11.2	9	8.4	11.2	10.2	4.2	5.2
γ_{11}	0.55	0.2	0.4	0.45	0.35	0.65	0.5	0.7	0.6	0.5	0.4	0.4
γ_{12}	0.45	0.8	0.6	0.55	0.65	0.35	0.5	0.3	0.4	0.5	0.6	0.6
γ_{21}	0.3	0.7	0.3	0.5	0.25	0.75	0.3	0.45	0.7	0.5	0.2	0.3
W_{max}	50	50	70	70	72	23	65	64	112	99	28	25
P_1	509	488	1017	828	947	790	1055	936	1797	1338	1408	1035
P_2	352	374	613	542	828	1352	664	712	1522	1228	156	151

6. Results and Discussion

In accordance with the objectives, the following results are obtained:

1. An optimization mathematical model of problem A is developed, the number of intact combat units of both sides at a given time is found (formulas (1)-(3)), the losses of side 2 are calculated depending on the first target distribution coefficient of side 1 and the largest losses of side 2 at a given time (formulas (4), (5)).

2. We describe the admissible set of values of the parameter K, namely the set U_{n_1,n_2} in which the maximum of the loss function of side 2 is found.

3. The optimization mathematical model of problem B, which is a generalization of problem A, is constructed in the form of system (6) with initial conditions (7); the admissibility of the corresponding parameters of the problem in the general case is described and explained for the case of $n = 2, m = 2$ (Fig. 6); the problem is solved for the case $n = 2,\ m = 2$, namely, the Cauchy problem (8)-(9), depending on the three cases, calculated in each case the number of intact combat units of both sides at a given time t_1, the function of losses of the second side depending on two parameters K_{11}, K_{21} is written out.

4. Computer models of the considered problems are built using Python programming. The corresponding solution algorithms are created (Fig.2, Fig. 7, Fig. 8).

5. The results of the corresponding numerical calculations are presented (Tabl. 1,2).

Note that the Cauchy problem (6)-(7) is solved individually depending on the number of heterogeneous combat units of sides 1,2, i.e., depending m and n. In this paper, we solve the problem (6)-(7) for the cases of $n = 1, m = 2$ and $n = 2, m = 2$. Of practical interest is the solution for the cases, for example $n = 1, m = 4$ and $n = 2, m = 3$, but this is possible in future works. The admissibility of the problem parameters is described in the general case, although it is shown only for the cases $n = 1, m = 3$ and $n = 2, m = 2$, because with larger values m and n permissible values the set of points of a unit $(m-1)n$-dimensional cube will be constrained, which is impossible to represent successfully.

Let us add practical applications and limitations of the models discussed.

1. The battlefield has become wider: rear areas can turn into combat zones; this requires additional targets to be taken into account in spatial models.

2. A “gray zone” (kill zones) has emerged, blurring the classic front line. As a result of the intensive use of unmanned systems, real-time reconnaissance, and

precision missile strikes, the traditional front line has been blurred. The impact of the "gray zone" can be modeled further by introducing spatial coefficients and stochastic parameters.

3. The role of heavy armored vehicles in certain conditions is reduced in favor of the high-precision use of UAVs and missiles; in the model, this is reflected by variables $\alpha_{ij}, i=1,2,...,n,\ j=1,2,...,m$ and β_{ji}, $j=1,2,...,m,\ i=1,2,...,n,$ as aggregate indicators of strike effectiveness.

4. The tactics of small mobile assault groups (combined with UAV support) mean that small units achieve significant strike effectiveness. The use of small mobile assault groups (motorized vehicles, light armored cars, etc.) is reflected in the models by small values of variable $N_i\ (i=1,2,...,n+m)$ and correspondingly large values β_{ji} and α_{ij}, $i=1,2,...,n,\ j=1,2,...,m$.

5. The mathematical and computer model considered in the form of a block diagram (Fig. 2) can be applied in the following cases, when side 1 has one type of weaponry, and side 2 has two types of weapons: side 1 (main battle tanks (MBT)), side 2 (mechanized infantry (IFV/armored personnel carriers) + self-propelled artillery/multiple launch rocket systems (MLRS)); side 1 (tactical missile systems), side 2 (air defense systems + reconnaissance/strike drones (small/medium types)); side 1 (motorized infantry (infantry units without armored vehicles)), side 2 (engineer and sapper units + barrel artillery batteries); side 1 (air strike units (attack aircraft/attack helicopters)), side 2 (two categories of air defense: long-range and short-range).

6. The mathematical and computer models considered in the form of a block diagram (Fig. 7, 8) can be applied in the cases when side 1 has two types of weapons and side 2 also has two types of weapons: side 1 (main battle tanks (MBT) + self-propelled artillery (SAU)), side 2 (mechanized infantry on BMP/BTR + empty anti-tank units (PTK/PTRK)); side 1 (strike UAVs (medium/heavy class) + ground reconnaissance units), side 2 (short-range air defense systems (portable and short-range) + medium-range anti-aircraft missile units); side 1 (attack helicopters/strike aircraft + reserve motorized infantry units), side 2 (self-propelled anti-tank missile systems/anti-tank batteries + mobile electronic warfare systems); side 1 (tactical missile units (short-range tactical missiles) + reconnaissance and targeting assets (radar/optical reconnaissance groups)), side 2 (air defense assets + protected command and control nodes); side 1 (airborne/assault units for raids + anti-tank infantry units (MANPADS/ATGM)), side 2 (reserve armored groups (light tanks/IFVs) + engineering and fortification units); side 1 (mechanized infantry + barrel artillery batteries), side 2 (mechanized infantry + multiple launch rocket systems).

7. Conclusions

As a result of the research and analysis, the following conclusions were made:

1) We have considered the optimal target allocation for two sides with different types of subdivisions, where each type of combat units directs fire at each type of enemy combat units in specific proportions. (with specific target distribution coefficients) in order to inflict the greatest number of casualties on enemy combat units at a given time moment. A mathematical optimization model is created as a system of differential equations (6) with initial conditions (7);

2) The objective function representing enemy losses is a function of (m-1)n variables, namely the target distribution coefficients of thethe party that distributes combat units in certain proportions;

3) The maximum value of the function of destroyed enemy units is found in the general case on a finite subset of nodes (m-1)n-dimensional cube.

It is found that the solution to the system of differential equations with initial conditions (6)-(7) depends on the values of m and n. The effectiveness of the model is illustrated by practically important examples with optimal allocation of the active side's resources for $n=1, m=2$ and $n=2, m=2$. The results using different initial data are presented in Tables 1 and 2. The specificity of the set of nodes, by the multitude of which the maximum of the function of destroyed enemy combat units is calculated, is explained by its structure for the case $n=1, m=3$. The constructed algorithms can be used to make well-founded decisions for predicting possible developments and for generalization to other values of n and m. Using the programs, we can predict the progress of the battle depending on time. In numerical calculations, the authors do not specify modern weapon types with their characteristics according to the problem formulation, since such data are not publicly available. However, based on the obtained formulas and solution algorithms, one can perform numerical calculations for a real combat situation of a "highly organized" battle, where the opposing sides have multiple types of combat resources (either $n=1, m=2$ or $n=2, m=2$).

The asymmetry of weapon types is taken into consideration with the introduction of the model with a multifactor effective rate-of-fire (velocity of impact) coefficient, which is not included in the classical Lanchester equations.

Future research directions can be related to:

- generalising the problem to account for the reinforcement of all units on both sides during the battle, including the deployment of reserves;
- generalising the problem to account for the rede-

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ployment of resources between areas of engagement during the battle.

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Conflict of Interest

The authors declare that they have no conflict of interest in relation to this research, whether financial, personal, author ship or otherwise, that could affect the research and its results presented in this paper.

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Data Availability

The manuscript has no associated data.

Use of Artificial Intelligence

The authors confirm that they did not use artificial intelligence methods while creating the presented work.

All authors have read and agreed to the publication of the finale version of this manuscript.

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МАТЕМАТИЧНЕ МОДЕЛЮВАННЯ БОЙОВИХ ДІЙ З ПЛАНУВАННЯМ КОЕФІЦІЄНТІВ ЦІЛЕРОЗПОДІЛУ ДЛЯ ОДНІЄЇ З ПРОТИБОРЧИХ СТОРІН

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Предметом дослідження є моделі бойових дій. **Метою** цього дослідження є побудова математичних і комп’ютерних моделей по виявленню оптимальних коефіцієнтів цілерозподілу активної сторони з метою завдати супротивнику найбільших втрат в заданий момент часу для випадку, коли сторони мають декілька видів озброєння і спрямовують вогонь по кожному виду бойових одиниць противника з деякими коефіцієнтами цілерозподілу. **Завдання:** 1) розглянути випадок, коли активна сторона має один вид озброєння, а інша сторона − два різнорідних види озброєння; 2) узагальнити задачу на випадок, коли активна сторона має n різнорідних видів озброєння, а інша сторона − m різнорідних видів озброєння; 3) провести детальне дослідження розв’язку задачі на випадок, коли сторони мають по два різнорідних види озброєння. Були отримані наступні **результати:** 1) побудовано математичну і комп’ютерну моделі для першого випадку завдання, проаналізовано допустимість параметрів задачі, наведено відповідні чисельні розрахунки; 2) створено математичну модель для другого випадку завдання, проаналізовано допустимість параметрів задачі 3) побудовано та досліджено математичну і комп’ютерну модель для третього завдання, наведено відповідні чисельні розрахунки. **Висновки.** Розглянуті в роботі приклади ілюструють, що, використовуючи комп’ютерну модель, можна прогнозувати, як розподілити бойові ресурси для боротьби з супротивником для досягнення успіху в бою в заданий момент часу, якщо супротивник має бойові одиниці двох видів. Задача має розв’язання і в загальному випадку, для цього потрібно дослідити на максимум функцію багатьох змінних в допустимих вузлах решітки, координатами яких є коефіцієнти цілерозподілу першої сторони. В роботі вказано як знайти ці допустимі вузли. Чисельні розрахунки до узагальнення задачі демонструють, що можна, змінюючи час, прогнозувати бій і в випадках, коли сторони мають декілька видів озброєння.

Ключові слова: оптимізаційна математична модель динаміки бою; бойові ресурси; швидкість ураження; коефіцієнт цілерозподілу; неушкоджені бойові одиниці; ударний потенціал.

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