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Solution of the First Fundamental Problem of Elasticity Theory for a Space with a Paraboloidal Notch and a Spherical Cavity

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The paper is devoted to the development of an analytical method for solving spatial problems of elasticity theory in domains with complex geometric shapes. The application of curvilinear coordinate systems – paraboloidal and spherical – to the investigation of the stress-strain state of elastic bodies is considered. The relevance of this research is determined by the necessity of mathematical modeling of the behavior of structures containing stress concentrators in the form of paraboloidal notches and spherical cavities, which are frequently encountered in engineering practice. Basic solutions of the Lamé vector equation describing the equilibrium of an isotropic elastic body are obtained in paraboloidal and spherical coordinates. Based on the proposed addition theorems, the relations between the basic solutions of the Lamé equation in paraboloidal and spherical coordinate systems are established. These relations allow for the transition from one representation to another when specifying boundary conditions on surfaces of various geometric shapes. The method proposed in the paper is applied to solve the first fundamental boundary value problem for an elastic space with a paraboloidal notch and a spherical cavity. The case where the paraboloidal surface is free from load and hydrostatic pressure acts on the spherical cavity is considered. The solution of the boundary value problem reduces to an infinite system of linear algebraic equations with respect to the unknown expansion coefficients. It is shown that the matrix coefficients of this system possess the property of exponential decay, which significantly simplifies the numerical implementation of the algorithm. A rigorous mathematical investigation of the properties of the resulting infinite system of equations is carried out. The complete continuity of the system operator in the Hilbert space of numerical sequences is proved under certain restrictions on the geometric parameters of the problem. The conditions for quasi-regularity and complete regularity of the infinite system are established, which proves the existence and uniqueness of the solution, as well as the possibility of obtaining it by the reduction method with any preassigned accuracy. The obtained results can be used for calculating the stress-strain state of structural elements of complex shape and are of interest to specialists in the fields of mechanics of deformable solids, applied mathematics, and computational methods.
Keywords: elasticity theory, axisymmetric problems, addition theorems, Lamé equation, boundary value problems, infinite systems.

Introduction

The investigation of the stress-strain state of elastic bodies with boundaries of complex geometric shapes remains one of the urgent problems in the mechanics of deformable solids. Of particular interest are problems for domains bounded by surfaces of several canonical coordinate systems, since it is in such structures that the main stress concentrations determining their strength characteristics arise. That is why, when considering bodies with paraboloidal or spherical boundaries, it is natural to apply curvilinear coordinate systems consistent with the geometry of the domain. The method of separation of variables, which has successfully proven itself in solving scalar boundary value problems of mathematical physics, can be extended to vector problems of elasticity theory. In this case, the key point is the construction of basic solutions of the Lamé vector equation in the corresponding coordinate systems. For paraboloidal coordinates, such solutions were obtained in [1]. Addition theorems for solutions of the Laplace equation in paraboloidal and spherical coordinates were

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obtained in [2]. The extension of these results to the vector case allows developing an effective method for solving boundary value problems for domains of the indicated type. In this case, the solution reduces to infinite systems of linear algebraic equations with exponentially decaying matrix coefficients. In the present paper, the application of addition theorems for basic solutions of the Lamé equation in paraboloidal and spherical coordinates, the development of an analytical method for solving mixed boundary value problems based on them, and the investigation of the properties of the resulting infinite systems of equations are considered.

1. Mathematical Apparatus of the Problem

Let the paraboloidal, cylindrical, and spherical coordinates be related by the equalities:

$$\rho = c\alpha\beta\ ,\ z = \frac{c}{2}(\alpha^2 - \beta^2)\ ;\quad \rho_1 = \rho\ ,\quad z_1 = z + h\ ;$$

$$\rho_1 = \rho\,,\ z_1 = z - H\ ,\ \rho_1 = r_1 \sin\theta_1\,,\ z_1 = r_1 \cos\theta_1$$

$$(0 \le \rho,\ \rho_1,\ r_1,\ \alpha,\ \beta < \infty;\ -\infty < z,\ z_1 < \infty\ ;\ h > 0),$$

where c is a dimensional factor ($c > 0$).

The equations $\alpha = const$, $\beta = const$ define two orthogonal families of paraboloids of revolution. Hereinafter, we assume that on the paraboloid $\beta = \beta_0$ $(\beta_0 = const)$.

The homogeneous equilibrium equations in displacements for an isotropic body reduce to the Lamé vector equation:

$$grad\ (div\ \bar{u}) + (1-2\nu)\Delta\bar{u} = 0, \tag{1.1}$$

where $\bar{u}$ is the displacement vector; ν is Poisson’s ratio.

Let us write the basic solutions of equation (1.1) in paraboloidal coordinates [1] in vector form:

$$\begin{aligned}
&\bar{U}_1^{+p}(\alpha,\beta;\lambda) = K_1(\lambda\beta)\, J_1(\lambda\alpha)\ \bar{e}_\rho - K_0(\lambda\beta)\, J_0(\lambda\alpha)\ \bar{e}_z\,,\\
&\bar{U}_2^{+p}(\alpha,\beta;\lambda) = 4\,(1-\nu)\ u_1^+\ \bar{e}_\rho + grad\ (-c\beta_0^2 u_0^+ - \rho\ u_1^+)\,,\\
&\bar{U}_1^{-p}\left(\alpha,\beta;\lambda\right) = I_1(\lambda\beta)\, J_1(\lambda\alpha)\ \bar{e}_\rho + I_0(\lambda\beta)\, J_0(\lambda\alpha)\ \bar{e}_z\,,\\
&\bar{U}_2^{-p}\left(\alpha,\beta;\lambda\right) = 4\,(1-\nu)\, u_1^-\, \bar{e}_\rho + grad\ (c\beta_0^2 u_0^- - \rho\, u_1^-)\,,\\
&u_m^+ = K_m(\lambda\beta)\, J_m(\lambda\alpha)\,,\ u_m^- = I_m(\lambda\beta)\, J_m(\lambda\alpha)\quad (m = 1,\ 2)\,,
\end{aligned} \tag{1.2}$$

where J_k are the Bessel functions of the first kind of order k $(k = \overline{0,\ 2})$;

I_k are the modified Bessel functions of the first kind of order k ;

K_k are modified Bessel functions of the second kind of order k .

Let us choose the basic solutions of equation (1.1) in spherical coordinates in the form of vector functions:

$$\begin{aligned}
&\bar{U}_{1,n}^{+s}(r_1,\theta_1) = c\ grad V_n^+ = -\left(\frac{c}{r_1}\right)^{n+2} \left[(n+1)\, P_n(\cos\theta_1)\ \bar{e}_{r_1} - P_n^1(\cos\theta_1)\ \bar{e}_{\theta_1}\right],\\
&\bar{U}_{2,n}^{+s}(r_1,\theta_1) = (z_1\ grad - \chi\,\bar{e}_z)\, V_{n-1}^+ - \frac{n+\chi}{2n-1}\, c\, V_{n-2}^+ =
\end{aligned} \tag{1.3}$$

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$$= -\left(\frac{c}{r_1}\right)^n \frac{1}{2n-1}\left[n\ (n+\chi)\ P_n(\cos\theta_1)\,\bar{e}_{r_1} - (n-1-\chi)\ P_n^1(\cos\theta_1)\,\bar{e}_{\theta_1}\right],$$

$$\bar{U}_{1,n}^{-s}\left(r_1,\theta_1\right) = c\ grad\,V_n^- = \left(\frac{r_1}{c}\right)^{n-1}\left[n\,P_n(\cos\theta_1)\,\bar{e}_{r_1} + P_n^1(\cos\theta_1)\ \bar{e}_{\theta_1}\right],$$

$$\bar{U}_{2,n}^{-s}\left(r_1,\theta_1\right) = \left(z_1\ grad - \chi\,\bar{e}_z\right)V_{n+1}^- - \frac{n+1-\chi}{2n+3}\,c\ grad\ V_{n+2}^- =$$

$$= \left(\frac{r_1}{c}\right)^{n+1}\frac{1}{2n+3}\left[(n+1)\ (n+1-\chi)\ P_n(\cos\theta_1)\,\bar{e}_{r_1} + (n+2+\chi)\ P_n^1(\cos\theta_1)\,\bar{e}_{\theta_1}\right],$$

$$V_j^+ = \left(\frac{c}{r_1}\right)^{j+1} P_j(\cos\theta_1)\ ,\quad V_j^- = \left(\frac{r_1}{c}\right)^{j} P_j(\cos\theta_1)\ \ (j = 1,\ 2)\ ,$$

$$\chi = 3 - 4\nu\,,$$

where $P_j^m(x)$ are the associated Legendre polynomials; $\bar{e}_\rho$, $\bar{e}_z$ ($\bar{e}_{r_1}$, $\bar{e}_{\theta_1}$) are the unit vectors of the cylindrical (spherical) coordinate system.

The solutions (1.2) and (1.3) are related by the relations [3]:

$$\bar{U}_{1,n}^{+s}(r_1,\theta_1) = \int_0^\infty \gamma_n^-(\lambda)\,\bar{U}_1^{-p}(\alpha,\beta;\lambda)\,d\lambda\,,$$

$$\bar{U}_{2,n}^{+s}(r_1,\theta_1) = \int_0^\infty \delta_n^-(\lambda)\,\bar{U}_2^{-p}(\alpha,\beta;\lambda)\,d\lambda + \int_0^\infty \omega_n^-(\lambda)\,\bar{U}_1^{-p}(\alpha,\beta;\lambda)\,d\lambda \quad \left(h > \frac{c\beta^2}{2}\right), \tag{1.4}$$

$$\bar{U}_1^{+p}(\alpha,\beta;\lambda) = \sum_{n=1}^\infty \eta_n^-(\lambda)\,\bar{U}_{1,n}^{-s}(r_1,\theta_1)\,,$$

$$\bar{U}_2^{+p}(\alpha,\beta;\lambda) = \sum_{n=0}^\infty \chi_n^-(\lambda)\,\bar{U}_{2,n}^{-s}(r_1,\theta_1) + \sum_{n=1}^\infty \xi_n^-(\lambda)\,\bar{U}_{1,n}^{-s}(r_1,\theta_1) \quad (r_1 < h)\,.$$

In the expansions (1.4), the following notations are introduced:

$$\gamma_n^-(\lambda) = -\frac{2\tau}{n!}\left(\frac{\lambda}{\tau}\right)^{n+2} K_{n+1}(\lambda\tau)\,,\quad \delta_n^-(\lambda) = -\frac{2\tau}{(n-1)!}\left(\frac{\lambda}{\tau}\right)^{n} K_{n-1}(\lambda\tau)\,,$$

$$\tau = \sqrt{2h/c}\,,$$

$$\omega_n^-(\lambda) = -\frac{2}{(n-1)!}\left(\frac{\lambda}{\tau}\right)^n\left[\frac{n\ (n+\chi)}{2n-1}\,\tau\,K_{n-1}(\lambda\tau) - \beta_0^2\,\lambda\,K_n(\lambda\tau)\right], \tag{1.5}$$

$$\eta_n^-(\lambda) = -\frac{1}{n!}\left(\frac{\lambda}{\tau}\right)^{n-1} K_{n-1}(\lambda\tau)\,,\quad \chi_n^-(\lambda) = -\frac{1}{(n+1)!}\left(\frac{\lambda}{\tau}\right)^{n+1} K_{n+1}(\lambda\tau)\,,$$

$$\xi_n^-(\lambda) = \frac{1}{n!}\left(\frac{\lambda}{\tau}\right)^{n-1}\left[\frac{(n-1)(n-1-\chi)}{2n-1}\,K_{n-1}(\lambda\tau) - \frac{\beta_0^2\,\lambda}{\tau}\,K_n(\lambda\tau)\right].$$

It can also be shown that the solutions (1.2) and (1.3) are related by the relations:

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$$\bar{U}_{1,n}^{+s}(r_1,\theta_1)=\int_0^\infty \gamma_n^+(\lambda)\ \bar{U}_1^{+p}(\alpha,\beta;\lambda)\,d\lambda\ ,$$

$$\bar{U}_{2,n}^{+s}(r_1,\theta_1)=\int_0^\infty \delta_n^+(\lambda)\,\bar{U}_2^{+p}(\alpha,\beta;\lambda)\ d\lambda+\int_0^\infty \omega_n^+(\lambda)\,\bar{U}_1^{+p}(\alpha,\beta;\lambda)\ d\lambda\ \left(H>-\frac{c\beta^2}{2}\right), \qquad (1.6)$$

$$\bar{U}_1^{-p}(\alpha,\beta;\lambda)=\sum_{n=1}^\infty \eta_n^+(\lambda)\,\bar{U}_{1,n}^{-s} r_1,\theta_1\ ,$$

$$\bar{U}_2^{-p}(\alpha,\beta;\lambda)=\sum_{n=0}^\infty \chi_n^+(\lambda)\ \bar{U}_{1,n}^{-s}(r_1,\theta_1)+\sum_{n=1}^\infty \xi_n^+(\lambda)\ \bar{U}_{1,n}^{-s}(r_1,\theta_1)\,.$$

The expressions for the quantities $\gamma_n^+(\lambda)$, $\delta_n^+(\lambda)$, $\omega_n^+(\lambda)$, $\eta_n^+(\lambda)$, $\chi_n^+(\lambda)$, $\xi_n^+(\lambda)$ are obtained by replacing the minus sign with a plus sign in the superscript in formulas (1.5), τ – with $t=\sqrt{2H/c}$ and $K_m(\lambda\tau)$ – with $(-1)^{m-1}J_m(\lambda t)$.

In deriving relations (1.4) and (1.6), the expansions of basic solutions of the Laplace equation in paraboloidal and spherical coordinates [2] were used:

$$\left(\frac{c}{r_1}\right)^{n+1} P_n(\cos\theta_1)=\frac{2\tau}{n!}\int_0^\infty\left(\frac{\lambda}{\tau}\right)^{n+1} K_n(\lambda\tau)\,I_0(\lambda\beta)\,J_0(\lambda\alpha)\,d\lambda \qquad \left(h>\frac{c\beta^2}{2}\right),$$

$$K_0(\lambda\beta)\,J_0(\lambda\alpha)=\sum_{n=0}^\infty\frac{1}{n!}\left(\frac{\lambda}{\tau}\right)^n K_n(\lambda\tau)\left(\frac{r_1}{c}\right)^n P_n(\cos\theta_1) \quad (r_1<h)\,,$$

$$\left(\frac{c}{r_1}\right)^{n+1} P_n(\cos\theta_1)=\frac{2\,(-1)^n}{n!}t\int_0^\infty\left(\frac{\lambda}{t}\right)^{n+1} J_n(\lambda t)\,K_0(\lambda\beta)\,J_0(\lambda\alpha)\,d\lambda \qquad \left(H>-\frac{c\beta^2}{2}\right),$$

$$I_0(\lambda\beta)\,J_0(\lambda\alpha)=\sum_{n=0}^\infty\frac{(-1)^n}{n!}\left(\frac{\lambda}{t}\right)^n J_n(\lambda t)\left(\frac{r_1}{c}\right)^n P_n(\cos\theta_1)\,,$$

$$t=\sqrt{2H/c}\,.$$

Relations (1.4) and (1.6), in conjunction with the Fourier method, can be applied to solving axisymmetric vector boundary value problems of elasticity theory in paraboloidal domains $0\le\beta\le\beta_0$, $\beta_0\le\beta<\infty$, weakened by a spherical cavity $0\le r_1\le R$. The implementation of such an approach leads to infinite systems of linear algebraic equations with exponentially decaying matrix coefficients.

2. Solution of the Boundary Value Problem for an Elastic Space Weakened by a Paraboloidal Notch and a Spherical Cavity

Let us outline the methodology for applying relations (1.4) and (1.6) using the example of solving the first fundamental problem for an elastic space weakened by a paraboloidal notch $0\le\beta\le\beta_0$ and a spherical cavity $0\le r_1\le R$, provided that $R<h-c\beta_0^2/2$. Let us restrict the analysis to the case where the surface of the paraboloid $\beta=\beta_0$ is traction-free, while the surface of the sphere $r_1=R$ is subjected to a hydrostatic pressure of intensity $\sigma_0>0$.

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The corresponding boundary conditions for the problem are:

$$\bar{F}\Big|_{\beta=\beta_0}=0 \quad , \quad \bar{F}\Big|_{r_1=R}=\sigma_0\,\bar{e}_{r_1}\,. \tag{2.1}$$

We shall express the general solution of the system of Lamé equations in the form:

$$\bar{u}=\int_0^\infty A_1(\lambda)\,\bar{U}_1^{+p}(\alpha,\beta;\lambda)\,d\lambda+\int_0^\infty A_2(\lambda)\,\bar{U}_2^{+p}(\alpha,\beta;\lambda)\;d\lambda+$$
$$+\sum_{n=0}^\infty B_n^{(1)}\bar{U}_{1,n}^{+s}(r_1,\theta_1)+\sum_{n=1}^\infty B_n^{(2)}\,\bar{U}_{2,n}^{+s}(r_1,\theta_1)\,.$$

If, on the basis of relations (1.4), the vector functions $\bar{U}_{1,n}^{+s}(r_1,\theta_1)$, $\bar{U}_{1,n}^{+s}(r_1,\theta_1)$, $\bar{U}_{2,n}^{+s}(r_1,\theta_1)$ are expanded in the vicinity of the paraboloid $\beta=\beta_0$, and the vector functions $\bar{U}_1^{+p}(\alpha,\beta;\lambda)$, $\bar{U}_2^{+p}(\alpha,\beta;\lambda)$ are expanded in the vicinity of the sphere $r_1=R$, and the corresponding stresses are computed, then, after satisfying the boundary conditions (2.1), we obtain the following relations between the integral densities $A_1(\lambda)$, $A_2(\lambda)$ and the coefficients $B_n^{(1)}$, $B_n^{(2)}$ of the series:

$$\begin{gathered}A_1(\lambda)\;f_1^+(\lambda)+A_2(\lambda)\;f_2^+(\lambda)+\;f_1^-(\lambda)\sum_{n=0}^\infty B_n^{(1)}\;\gamma_n^-(\lambda)+\\+f_2^-(\lambda)\sum_{n=1}^\infty B_n^{(2)}\;\delta_n^-(\lambda)+\;f_1^-(\lambda)\sum_{n=1}^\infty B_n^{(2)}\;\omega_n^-(\lambda)=0\;,\\A_1(\lambda)\;g_1^+(\lambda)+A_2(\lambda)\;g_2^+(\lambda)+g_1^-(\lambda)\sum_{n=0}^\infty B_n^{(1)}\;\gamma_n^-(\lambda)+\\+g_2^-(\lambda)\sum_{n=1}^\infty B_n^{(2)}\;\delta_n^-(\lambda)+g_1^-(\lambda)\sum_{n=1}^\infty B_n^{(2)}\;\omega_n^-(\lambda)=0\;.\end{gathered}\tag{2.2}$$

$$\begin{gathered}B_0^{(1)}\left(\frac{c}{R}\right)^3+\frac{1+\nu}{3}\,D_0^{(2)}=-\frac{\sigma_0\,c}{4G}\,,\\B_k^{(1)}(k+1)(k+2)\left(\frac{c}{R}\right)^{k+2}+B_k^{(2)}\,\frac{k}{2k-1}(k^2+3k-2\nu)\left(\frac{c}{R}\right)^k=\\=-D_k^{(1)}\,k\,(k-1)\left(\frac{R}{c}\right)^{k-1}-D_k^{(2)}\,\frac{k+1}{2k+3}\,(k^2-k-2-2\nu)\left(\frac{R}{c}\right)^{k+1},\\-B_k^{(1)}\,(k+2)\left(\frac{c}{R}\right)^{k+2}-B_k^{(2)}\,\frac{k^2-2+2\nu}{2k-1}\left(\frac{c}{R}\right)^k=\\=-D_k^{(1)}(k-1)\left(\frac{R}{c}\right)^{k-1}-D_k^{(2)}\,\frac{k^2+2k-1+2\nu}{2k+3}\left(\frac{R}{c}\right)^{k+1}.\end{gathered}\tag{2.3}$$

For convenience, we introduce the following notation:

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$$D_k^{(1)} = \int_0^\infty A_1(\lambda)\,\eta_k^-(\lambda)\,d\lambda + \int_0^\infty A_2(\lambda)\,\xi_k^-(\lambda)\,d\lambda\,, \quad D_k^{(2)} = \int_0^\infty A_2(\lambda)\,\chi_k^-(\lambda)\,d\lambda\,,$$

$$f_1^+(\lambda) = -\lambda\beta_0\,K_0(\lambda\beta_0) - K_1(\lambda\beta_0)\,, \quad g_1^+(\lambda) = \lambda\beta_0\,K_1(\lambda\beta_0)\,,$$

$$f_2^+(\lambda) = (2\nu-3)\lambda\beta_0\,K_0(\lambda\beta_0) - [4(1-\nu)+\lambda^2\beta_0^2]\,K_1(\lambda\beta_0)\,,$$

$$g_2^+(\lambda) = 2(1-\nu)\lambda\beta_0\,K_1(\lambda\beta_0) + \lambda^2\beta_0{}^2 K_0(\lambda\beta_0)\,,$$

$$f_1^-(\lambda) = \lambda\beta_0\,I_0(\lambda\beta_0) - I_1(\lambda\beta_0)\,, \quad g_1^-(\lambda) = \lambda\beta_0\,I_1(\lambda\beta_0)\,,$$

$$f_2^-(\lambda) = (3-2\nu)\lambda\beta_0\,I_0(\lambda\beta_0) - [4(1-\nu)+\lambda^2\beta_0^2]\,I_1(\lambda\beta_0)\,,$$

$$g_2^-(\lambda) = 2(1-\nu)\lambda\beta_0\,I_0(\lambda\beta_0) - \lambda^2\beta_0{}^2\,I_0(\lambda\beta_0)\,.$$

Let us consider new unknowns $b_k^{(1)}$, $b_k^{(2)}$:

$$(2k+1)\,B_k^{(1)}\left(\frac{c}{R}\right)^{k+3} = -\frac{\sigma_0\,c}{4G}\,b_k^{(1)}\,, \quad k\,B_k^{(2)}\left(\frac{c}{R}\right)^{k+1} = \frac{\sigma_0\,c}{4G}\,b_k^{(2)}\,.$$

Eliminating the functions $A_1(\lambda)$, $A_2(\lambda)$ from equations (2.2) and (2.3), we obtain that $b_1^{(2)} = 0$, $B_1^{(2)} = 0$, and an infinite system of linear algebraic equations:

$$b_k^{(1)} = \sum_{n=0}^\infty D_{kn}^{(11)}\,b_n^{(1)} + \sum_{n=2}^\infty D_{kn}^{(12)}\,b_n^{(2)} + f_k^{(1)} \quad (k=0,\,1,\,\ldots)\,,$$

$$b_k^{(2)} = \sum_{n=0}^\infty D_{kn}^{(21)}\,b_n^{(1)} + \sum_{n=2}^\infty D_{kn}^{(22)}\,b_n^{(2)} \quad (k=2,3,\ldots)\,, \tag{2.4}$$

$$f_0^{(1)} = 1\,, \quad f_k^{(1)} = 0 \quad (k=1,2,\ldots)\,,$$

$$D_{kn}^{(2j)} = \frac{k\,(k-1)}{2\,\Delta_k}\,\varepsilon^{k+n-1}\alpha_n^{(j)}\left[(2k+1)\,T_{kn}^{(2j)} + (k+1)\varepsilon^2\,S_{kn}^{(2j)}\right],$$

$$D_{kn}^{(1j)} = D_{kn}^{(2j)} + \omega_k\,\alpha_n^{(j)}\,\varepsilon^{k+n+1} S_{kn}^{(2j)} \quad (j=1,2)\,,$$

$$\varepsilon = R/c\,, \quad \alpha_n^{(1)} = \varepsilon^2/(2n+1)\,, \quad \alpha_n^{(2)} = -1/n\,,$$

$$\bar{\omega}_k = 2\,\frac{k^2+2k\nu+k+1+\nu}{(k+2)(2k+3)}\,, \quad \Delta_k = \frac{k^2-2k\nu+k+1-\nu}{2k-1}\,,$$

$$T_{kn}^{(21)} = -2\,\frac{\tau^{-k-n}}{k!\;n!}\int_0^\infty \gamma_n(\lambda)\left[\xi_k(\lambda) + \eta_k(\lambda)\,\Delta_1(\lambda)\right]\frac{d\lambda}{\Delta(\lambda)}\,,$$

$$T_{kn}^{(22)} = -2\,\frac{\tau^{-k-n+2}}{k!\,(n-1)!}\left\{\int_0^\infty \xi_k(\lambda)\left[\omega_n(\lambda) + \delta_n(\lambda)\,\Delta_2(\lambda)\right]\frac{d\lambda}{\Delta(\lambda)} - \right.$$

$$\left. - \int_0^\infty \eta_k(\lambda)\left[\omega_n(\lambda)\,\Delta_1(\lambda) - \delta_n(\lambda)\,\Delta_3(\lambda)\right]\frac{d\lambda}{\Delta(\lambda)}\right\}\,,$$

$$S_{kn}^{(21)} = \frac{2\,\tau^{-k-n}}{(k+1)!\;n!}\int_0^\infty \frac{\chi_k(\lambda)\;\gamma_n(\lambda)}{\Delta(\lambda)}\,d\lambda\,,$$

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$$\delta_{kn}^{(22)} = 2\,\frac{\tau^{-k-n}}{(k+1)!\,(n-1)!}\int_0^\infty \chi_k(\lambda)\left[\omega_n(\lambda)+\delta_n(\lambda)\,\Delta_2(\lambda)\right]\frac{d\lambda}{\Delta(\lambda)},$$

$$\gamma_n(\lambda)=\lambda^{n+2}K_{n+1}(\lambda\tau),\quad \delta_n(\lambda)=\lambda^n K_{n-1}(\lambda\tau),$$

$$\omega_n(\lambda)=\lambda^n\left[\frac{n\,(n+\chi)}{2n-1}\,K_{n-1}(\lambda\tau)-\frac{\beta_0{}^2\lambda}{\tau}\,K_n(\lambda\tau)\right],$$

$$\eta_k(\lambda)=\lambda^{k-1}K_{k-1}(\lambda\tau),\quad \chi_k(\lambda)=\lambda^{k+1}K_{k+1}(\lambda\tau),$$

$$\xi_k(\lambda)=\lambda^{k-1}\left[\frac{(k-1)(k-1-\chi)}{2k-1}\,K_{k-1}(\lambda\tau)-\frac{\beta_0^2\,\lambda}{\tau}\,K_k(\lambda\tau)\right],$$

$$\Delta(\lambda)=2(1-\nu)\,K_1^2(\lambda\beta_0)+\lambda^2\beta_0^2\,[K_1^2(\lambda\beta_0)-K_0^2(\lambda\beta_0)],$$

$$\Delta_2(\lambda)+2-2\nu=2\nu-2-\Delta_1(\lambda)=2\,(1-\nu)\,K_1(\lambda\beta_0)\,I_1(\lambda\beta_0)+$$
$$+\lambda^2\beta_0^2\,\left[K_1(\lambda\beta_0)\,I_1(\lambda\beta_0)+K_0(\lambda\beta_0)\,I_0(\lambda\beta_0)\right],$$

$$\Delta_3(\lambda)=\lambda^2\beta_0^2-2\,(1-\nu)(1-2\nu).$$

3. Investigation of the Infinite System of Linear Algebraic Equations

We investigate the properties of the infinite system (2.4). To this end, we prove that its operator is completely continuous in l_2 for $R < h - c\beta_0^2/2$.

Based on the chain of inequalities [4]:

$$K_1(x) > K_0(x),\quad I_0(x) > I_1(x),$$

$$I_0(x)K_0(x) < I_0(x)K_1(x) < I_0(x)K_1(x) + I_1(x)K_0(x) = x^{-1},$$

$$I_1(x)K_1(x) < I_0(x)K_1(x) < x^{-1},$$

$$K_1(x)=\int_0^\infty e^{-x\,cht}cht\,dt > \int_0^\infty e^{-x\,cht}sht\,dt = x^{-1}e^{-x},$$

$$K_1^{-2}(x) < x^2e^{2x} < 2x^2ch2x$$

we have the following estimates:

$$\frac{1}{\Delta(\lambda)} < \frac{\beta_0^2}{1-\nu}\lambda^2 ch2\lambda\beta_0,$$
$$\left|\Delta_j(\lambda)\right| \le 2(1-\nu)\left(1+\frac{1}{\lambda\beta_0}\right)+2\lambda\beta_0 \quad (j=1,2). \tag{3.1}$$

Considering the structure of the matrix coefficients $D_{kn}^{(ij)}$ and inequalities (3.1), the investigation of the properties of the infinite system (2.4) is reduced to the estimation of integrals of the form:

$$\overline{R}_{lm} = \int_0^\infty \lambda^{l+m+s}K_l(\lambda\tau)K_m(\lambda\tau)ch\,2\lambda\beta_0\,d\lambda = \tau^{-l-m-s-1}R_{lm},$$

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$$R_{lm} = \int_0^\infty x^{l+m+s} K_l(x) K_m(x)\, ch\left(\frac{2\beta_0}{\tau} x\right) dx$$

$$(l = 0, 1, 2, \ldots,\ m = 0, 1, 2, \ldots,\ s = \overline{1, n}).$$

Using the Cauchy–Schwarz inequality:

$$R_{lm}^2 \le R_{ll} R_{mm}, \quad R_{jj} = \int_0^\infty x^{2j+s} K_j^2(x)\, ch\left(\frac{2\beta_0}{\tau} x\right) dx,$$

and the integral representation [4]:

$$K_\mu(X) K_\nu(x) = \int_{-\infty}^{\infty} e^{-(\mu-\nu)t} \left(\frac{Xe^t + xe^{-t}}{Xe^{-t} + xe^t}\right)^{1/2(\mu+\nu)} K_{\mu+\nu}(u) dt,$$

$$u = \sqrt{X^2 + x^2 + 2X \cdot x \cdot ch2t}, \quad X > 0, \quad x > 0,$$

we consequently obtain:

$$R_{jj} = 2\int_0^\infty dt \int_0^\infty x^{2j+s} K_{2j}(2xcht)\, ch\left(\frac{2\beta_0}{\tau} x\right) dx,$$

$$R_{jj} < 2\int_0^\infty dt \int_0^\infty x^{2j+s} K_{2j+s}(2xcht)\, ch\left(\frac{2\beta_0}{\tau} x\right) dx \quad (s = \overline{1, n}).$$

Based on the equality [5]:

$$\int_0^\infty x^\nu K_\nu(cx) \cos bx dx = 2^{\nu-1}\sqrt{\pi} c^\nu \Gamma(\nu + 1/2)(b^2 + c^2)^{-\nu-1/2},$$

where $\Gamma(x)$ is the Gamma function, for $\nu = 2j + s$, $b = 2i\beta_0/\tau$ and $c = ch2t$, we obtain:

$$R_{jj} < \frac{\sqrt{\pi}}{2} \Gamma(2j + s + 1/2) \int_0^\infty \frac{ch^{2j+s} t dt}{(ch^2 t - \beta_0^2 \tau^{-2})^{2j+s+1/2}}.$$

By setting $ch^2 t - \beta_0^2 \tau^{-2} = u(1 - \beta_0^2 \tau^{-2})$, we rewrite this inequality as follows:

$$R_{jj} < \frac{\sqrt{\pi}\Gamma(2j + s + 1/2)}{4(1 - \beta_0^2 \tau^{-2})^{2j+s}} \int_1^\infty \frac{\left[\beta_0^2 \tau^{-2} + (1 - \beta_0^2 \tau^{-2})u\right]^{j+s/2-1/2}}{u^{2j+s+1/2}\sqrt{u-1}} du.$$

Since $u > c\beta_0^2/2$, then $0 < \beta_0^2 \tau^{-2} + (1 - \beta_0^2 \tau^{-2})\, u \le u$. Therefore:

$$R_{jj} < \frac{\sqrt{\pi}\Gamma(2j + s + 1/2)}{4(1 - \beta_0^2 \tau^{-2})^{2j+s}} \int_1^\infty \frac{du}{u\sqrt{u-1}} = \frac{\pi^{3/2}\Gamma(2j + s + 1/2)}{4(1 - \beta_0^2 \tau^{-2})^{2j+s}} < \frac{\pi^{3/2}(2j + s)!}{4(1 - \beta_0^2 \tau^{-2})^{2j+s}}.$$

It follows that:

$$\left| D_{kn}^{(ij)} \right| \le M \frac{\Gamma(n + k + 1)}{(2\beta^2)^{n+k+1}} R^k \quad (M = const,\ M > 0). \tag{3.2}$$

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With the aid of estimate (3.2), it is readily verified that, for $R < h - c\beta_0^2 / 2$ $(\tau^2 - \beta_0^2 > 2\varepsilon)$, the following inequalities hold:

$$\sum_{(k,\, n)}^{\infty} \left| D_{kn}^{(ij)} \right|^2 < \infty \quad (i, j = 1, 2) . \tag{3.3}$$

Inequality (3.3) and the fact that $\{ f_k^{(i)} \}$ belongs to the Hilbert space of numerical sequences l_2 imply, that the operator of the infinite system (2.4) is completely continuous from l_2 into l_2 for $R < h - c\beta_0^2 / 2$. It follows that for almost all values of the parameter $\varepsilon_0 = R(h - c\beta_0^2 / 2)^{-1} \in (0,1)$, the solution of the infinite system (2.4) in l_2 exists, is unique, and can be found using the reduction method [5]. Since the corresponding homogeneous boundary value problem has only the trivial solution, this conclusion is valid for all $\varepsilon_0 \in (0,1)$. Estimates (3.2) also allow us to conclude that the infinite system (2.4) is quasi-regular for $0 < \varepsilon_0 < 1$ and fully regular for $0 < \varepsilon_0 \le \bar{\varepsilon}_0$ for some $\bar{\varepsilon}_0 \in (0,1)$, while the matrix coefficients $D_{kn}^{(ij)}$ decay exponentially for $k + n \to \infty$. The constraint $0 < \varepsilon_0 < 1$ on the possible values of the parameter ε_0 is naturally associated with the formulation of the problem under consideration and implies that the sphere $r_1 = R$ and the paraboloid $\beta = \beta_0$ neither intersect nor touch each other.

Conclusions

1. An analytical method is proposed for solving three-dimensional problems of elasticity theory for domains of complex geometric shape bounded by intersecting coordinate surfaces of the paraboloidal and spherical coordinate systems.
2. Basic solutions of the vector Lamé equation are constructed in paraboloidal and spherical coordinates, describing the equilibrium of an isotropic elastic body.
3. Formulas for re-expansion (addition theorems) linking the basic solutions of the Lamé equation in paraboloidal and spherical coordinate systems are derived to enable the exact satisfaction of boundary conditions on surfaces of different shapes.
4. The first fundamental boundary value problem for an elastic space weakened by a paraboloidal notch and a spherical cavity has been solved. The specific case considered involves a traction-free paraboloidal surface and hydrostatic pressure on the cavity, reducing the problem to an infinite system of linear algebraic equations.
5. A rigorous mathematical investigation proved the complete continuity of the system's operator in the Hilbert space (under geometric constraints) and established quasi- and full regularity, theoretically justifying the existence, uniqueness, and applicability of the reduction method with arbitrarily prescribed accuracy.

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Розв'язання першої основної задачі теорії пружності для простору з параболоїдальним вирізом та сферичною порожниною

Роботу присвячено розробці аналітичного методу розв'язання просторових задач теорії пружності в областях складної геометричної форми. Розглянуто застосування криволінійних систем координат – параболоїдальної та сферичної – до дослідження напружено-деформованого стану пружних тіл. Актуальність такого дослідження обумовлена необхідністю математичного моделювання поведінки конструкцій з концентраторами напружень у вигляді параболоїдальних вирізів і сферичних порожнин, які часто трапляються в інженерній практиці. Отримано базисні розв'язки векторного рівняння Ламе, що описує рівновагу ізотропного пружного тіла, у параболоїдальних і сферичних координатах. На основі запропонованих теорем додавання встановлено співвідношення між базисними розв'язками рівняння Ламе у параболоїдальній та сферичній системах координат. Ці співвідношення дозволяють переходити від одного подання до іншого у разі задання граничних умов на поверхнях різної геометричної форми.

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Запропонований у роботі метод застосовано до розв'язання першої основної граничної задачі для пружного простору з параболоїдальним вирізом і сферичною порожниною. Розглянуто випадок, коли параболоїдальна поверхня вільна від навантаження, а на сферичній порожнині діє гідростатичний тиск. Розв'язання крайової задачі зведено до нескінченної системи лінійних алгебраїчних рівнянь відносно невідомих коефіцієнтів розкладу. Показано, що матричні коефіцієнти цієї системи експоненційно спадають, що суттєво спрощує чисельну реалізацію алгоритму. Проведено строге математичне дослідження властивостей здобутої нескінченної системи рівнянь. Доведено повну неперервність оператора системи у гільбертовому просторі числових послідовностей за певних обмежень на геометричні параметри задачі. Встановлено умови квазірегулярності та повної регулярності нескінченної системи, що доводить існування та єдиність розв'язку, а також можливість його знаходження методом редукції з будь-якою наперед заданою точністю. Отримані результати можуть бути використані для розрахунку напружено-деформованого стану елементів конструкцій складної форми і становлять інтерес для фахівців у галузі механіки деформівного твердого тіла, прикладної математики та обчислювальних методів.

Ключові слова: теорія пружності, осесиметричні задачі, теореми додавання, рівняння Ламе, граничні задачі, нескінченні системи.

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