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Pavlo LUKIANOV, Lin SONG

National Aviation University, Kyiv, Ukraine

UNSTEADY INCOMPRESSIBLE LAMINAR BOUNDARY LAYER: TIME AND SPACE VARIABLE MOLECULAR VISCOSITY

The subject *of this work is two approaches to describe the laminar unsteady flow of an incompressible fluid in the boundary layer. In the first approach, the viscosity of the fluid and the acceleration with which the plane is set in motion are considered constant. In essence, this is Rayleigh's problem. The solution obtained on the basis of these assumptions asymptotically converges to the well-known self-similar Stokes solution. It is important that the solutions of Stokes and Rayleigh asymptotically at large values of time correspond to the disappearance of shear stresses between the liquid and moving plane after acceleration. A paradox emerges the equations derived by Stokes to describe internal friction indicate the absence of the same friction between a moving body and fluid. Since research using the calculus of variation methods revealed that the molecular viscosity inside the stationary boundary layer should depend on the distance to the moving surface, the corresponding non-steady problem was considered. As a result, as before for the steady case, solutions describing both non-gradient and gradient flows of incompressible fluid in the boundary layer are obtained. The asymptotic analysis of the transition to steady flow testifies the consistency of these solutions. For the case of non-gradient flow, a comparison of the classical solution with the solution corresponding to the extreme fluid flow rate carried by the moving surface is made. It is shown that according to the solution obtained on the basis of the calculus of variation approach, the shear stress on the surface does not disappear anywhere after the motion becomes steady but, as expected, acquires a constant value.* ***The research methods*** *are purely theoretical and the results are analyzed by comparison with available theoretical and experimental data and compliance with the fundamental laws of physics, in particular the law of conservation of energy. These methods are based on the construction of analytical mathematical models, which are differential equations in partial derivatives supplemented with appropriate physical initial and boundary conditions. In addition, Euler's differential equations for the extreme of functional theory are used (in this paper, this is the extreme of fluid flow rate across the cross-section of the boundary layer). When solving these equations, the well-known Fourier method of variable separation is used. Arbitrary functions of time arising during partial integration (by one of the variables – the spatial coordinate) are determined from the conditions of asymptotic convergence of the solutions of non-steady problems to the corresponding solutions of steady problems.* ***Conclusions.*** *The presented results are of fundamental importance for understanding the physics of the flow around aircraft parts, as they indicate the contradiction of the existing idea of the reversibility of direct and inverse problems: the motion of a body in a still fluid and the flow of a fluid around an immobile body.*

Keywords: *aircraft; laminar boundary layer; unsteady incompressible flow; variable molecular viscosity.*

Introduction

Regardless of whether the motion of a body in a fluid is steady or unsteady, it is always affected by the force of friction. This, of course, also applies to the motion of aircraft. The frictional force of the fluid on the surface of the aircraft is created in a rather thin boundary layer - in the immediate vicinity of the streamlined surface [1].

The boundary layer is an important component of optimizing the aerodynamic characteristics of the wing profile [2], improving the flow around the blade and increasing its non-skid properties [3]. The flow in the boundary layer is also related to the formation of vortex tracks, which is very important for flight safety, especially in non-stationary regimes - during take-off and landing [4].

In the middle of the 19th century, one of the new problems at that time was the study of the influence of the resistance of the environment on the motion of the pendulum. Scientists did not abandon the idea of creating a so-called "eternal engine". Ideas, as you know, do not arise from nothing. They (these ideas) are the result of certain theories. One of these theories is the Stokes model of viscous fluid motion [5]. According to this model, in case of non-steady flow, under the assumption of constant molecular viscosity, a false conclusion is reached about the possibility of the existence of a perpetual motion machine. To point out the shortcomings of Stokes' theory, this paper considers two problems: the main problem deals with acceleration and subsequent steady motion, and the second, auxiliary problem, deals with unsteady fluid flow along a fixed plane.

Despite the fact that today almost all the efforts of researchers are directed to the study of turbulent flows, nevertheless, the theory of the non-steady laminar boundary layer has not yet been completed. This becomes clear after studying the works of Stokes [5, 6]. Only recently, it was possible, from unified position, to obtain an analytical description of steady gradient and non-gradient incompressible flows in the boundary layer [7].

As for the non-steady boundary layer, various problems were considered. First of all, we should mention the general solution for an arbitrary law of acceleration of a plane to a finite speed, obtained by Stokes (see [6], (185)) and its special case for instantaneous motion of a plane with a constant speed (see [6], (186)). Apparently, aware of the physical impossibility of instantaneous acceleration of a plane, like any other body of finite mass, to a finite speed, Rayleigh considered the case of uniformly accelerating an infinite plane from a state of rest into motion at a constant speed [8]. The Stokes solution obtained on the basis of the constancy of viscosity (the Navier-Stokes equation) and its special case considered by Rayleigh (see [8], formulas (17), (18)) indicate a discrepancy with physics. Thus, in the work of Rayleigh [8] it is indicated (see the last formula of the third paragraph of the cited work) that the velocity gradient, and with it, taking into account the constancy of the molecular viscosity, the shear stress τ asymptotically decrease in time according to the law

$$\mu \frac{\partial V_x}{\partial y}\bigg|_{y=0} = \tau(t >> t_b)\, t^{-1/2} \rightarrow 0,\ t \rightarrow \infty. \qquad (1)$$

In (1) t_b is the acceleration time,

V_x is the speed of the plane,

μ is a molecular viscosity,

t,y are time and coordinate normal to the plane in accordance.

It immediately follows from (1) that with steady motion, the motion resistance is zero. In other words, we accelerate the body, and it continues to move, without supplying energy from the outside, at a constant speed. Of course, this is completely wrong, as is trying to create a perpetual motion machine.

The following considerations will help us, firstly, to make sure that in the problem of the motion of a plane in space, the velocity cannot be constant or increase anywhere, and, secondly, they will indicate the unphysical correspondence of (1). It is clear that when a plane moves, the product of the velocity and the viscous shear stress on the surface of this plane is nothing but the power that the plane transmits to the surrounding space. If the motion is steady, then the mentioned power should disappear somewhere every time moment. So it is: it disappears due to viscous dissipation (heating of the liquid is not taken into account):

$$\begin{aligned} & V_x(y=0)\cdot\tau(y=0)= \\ & = -\int_0^\infty \mu \frac{dV_x}{dy}\frac{dV_x}{dy}dy = -\tau(0)\int_0^\infty \frac{dV_x}{dy}dy = \\ & = -\tau(y=0)\big(V_x(y=\infty)-V_x(y=0)\big). \end{aligned} \qquad (2)$$

When deriving (2), it was taken into account that $\mu(dV_x/dy)=\tau(y=0)=Const$, which corresponds to a non-gradient flow. Therefore, the balance of these capacities is possible under the condition that $V_x(y=\infty)=0$. This is an important fact because it prevents the velocity field from having a constant value at infinity and everywhere. On the other hand, the shear stress is constant in time (and in the case of acceleration of the body, it asymptotically approaches a constant value). Therefore, expression (1), being absolutely correct mathematically, has nothing to do with the real physics of the problem, since energy dissipation occurs at every moment of time.

These contradictions disappear when, in the generalized Navier-Stokes equations, the viscosity in the boundary layer is considered, in the general case, as a variable: in steady motion, the viscosity is a function of the distance to the boundary of the solid body (wall) (see [7]), and for non-steady motion is a function of time and distance (see below).

The development of this topic can be found in subsequent works by Gohrtler [9], Howards [10], Sowersby [11, 12], and Watson [13], in which the growth of the boundary layer and the boundary layer in a semi-infinite region of various shapes are considered. Flows in the Stokes boundary layer in the form of harmonic oscillations are still used now when considering various non-steady problems [14].

For a better understanding of the further presentation, let's briefly review the article [7], devoted to the steady boundary layer. The starting point of the study is Schlichting's monograph [15], where the reader can find a summary of the problem (at the time of publication of this book). An important role is assigned to experimental work on measuring the boundary layer [16]. The theory of the boundary layer, which is currently used, was developed by Prandtl [17] almost sixty years after the publication of the work of Stokes [5]. Isn't this a paradox? The Stokes theory [5] of the motion of a viscous fluid already exists, but its purpose, which is primarily related to the description of internal friction in the boundary layer, finds its implementation only in the work of Prandtl [17]. The answer to this question can be found

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in [7]. The theory of the steady boundary layer of an incompressible flow was further developed in the works of Blasius [18] (as well as a generalization of his problem [19]) and Boltze [20], as well as in the works of Karman [21] and Pohlhausen [22]. Van Drist's work [23], which is somewhat separated in time, although formally dedicated to the turbulent boundary layer, uses an exponential multiplier that is characteristic of the laminar boundary layer [6]. Successes in the study of the boundary layer before 1970 are described in Loitsianskyi's review [24]. Modern works on the laminar boundary layer include the formally mathematical works of Wyburn [25, 26], the work of Sohrab [27], based on a statistical description of the flow physics in both laminar and turbulent boundary layers. And, perhapsy, the work of Abdullah Grafor [28], in which the ideas of the Polhausen method [22] are developed, also deserves attention.

1. Formulation of the problem

In this paper, we consider unsteady incompressible flows in laminar boundary layers:

– a non-gradient non-steady boundary layer formed during the acceleration motion of an infinite plane;

– a gradient non-steady boundary layer, which is formed when a fluid flows around along fixed plane at a constant speed.

The purpose of the work is to obtain, based on the calculus of variation approach, analytical distributions for the velocity field in gradient and non-gradient laminar unsteady boundary layers of an incompressible fluid and compare them with the classical ones, pointing out the shortcomings of the latter.

2. Laminar unsteady incompressible fluid flow due to uniform acceleration of the plane

It is impossible to instantly accelerate the body to a finite speed: an infinitely large power is required. Therefore, no matter how small the acceleration time of a rocket or projectile is, it is still finite. The plane accelerates or decelerates within the time limit. This tells us that the problem of a laminar boundary layer that is constantly changing over a finite time is quite real. Apparently, that is why Rayleigh, as stated in the introduction, solved the problem of uniform acceleration of a plane to a constant speed [8]. What follows in this section cannot be considered entirely original. It is rather a bridge between the classical (old) presentation and the modern one, which is becoming more and more difficult to understand every day. Although the formal mathematical notation and representation differ from the works of Stokes [6] and Rayleigh [8], it is essentially the same physical problem. In this work, the problem of braking will not be considered: only acceleration.

The speed of uniform acceleration of the aircraft for finite time τ_b and subsequent steady motion $(y=0)$ is described by law [29]

$$V_{x|y=0} = U_0\left[\frac{t}{\tau_b}H(t) - \frac{t-\tau_b}{\tau_b}H(t-\tau_b)\right]. \quad (3)$$

In (3) $H(t)$ is the Heaviside function. The second boundary condition was already mentioned above:

$$V_x(y=\infty)=0.$$

Solution of the Navier-Stokes equation

$$\frac{\partial V_x}{\partial t} = \nu\frac{\partial^2 V_x}{\partial y^2}$$

with the listed boundary conditions and the initial condition (which is automatically fulfilled in (3)) has the form (compare with the solutions of Stokes [6], (formulas 185, 186) and Rayleigh [8], formulas 17, 18):

$$V_x(y,t) = \frac{U_0 y}{2\sqrt{\nu\pi}}\int_0^t \frac{\exp\left[-y^2/(4\nu(t-\tau_b))\right]}{(t-\tau)^{3/2}}\times \left(\frac{\tau}{\tau_b}H(\tau) - \frac{\tau-\tau_b}{\tau_b}H(\tau-\tau_b)\right)d\tau. \quad (4)$$

The integral of the right-hand side of (4) can be conveniently expanded into the sum of the following two integrals:

$$I_1(y,t) = \frac{U_0 y}{2\sqrt{\nu\pi}}\int_0^{\tau_b} \frac{\exp\left[-y^2/(4\nu(t-\tau_b))\right]}{(t-\tau)^{3/2}}\frac{\tau}{\tau_b}d\tau,$$

$$I_2(y,t) = \frac{U_0 y}{2\sqrt{\nu\pi}}\int_{\tau_b}^{t} \frac{\exp\left[-y^2/(4\nu(t-\tau_b))\right]}{(t-\tau)^{3/2}}d\tau.$$

These integrals are respectively equal to $I_1(y,\tau_p)-I_1(y,0)$ and $I_2(y,t)-I_2(y,\tau_p)$, where

$$I_1(y,t) = \frac{U_0 y}{\tau_b 2\nu\sqrt{\pi}}\left\{2\sqrt{t-\tau_b}\exp\left[-\frac{1}{4}\frac{y^2}{(t-\tau_b)}\right] + \left(y\sqrt{\pi} + \frac{2t\sqrt{\pi\nu}}{y}\right)\operatorname{erf}\left[\frac{1}{2}\frac{y}{\sqrt{\nu(t-\tau_b)}}\right]\right\},$$

$$I_2(y,t)=\frac{U_0}{\tau_b\sqrt{\nu}}\text{erf}\left[\frac{1}{2}\frac{y}{\sqrt{\nu(t-\tau_b)}}\right]. \quad (5)$$

It is easy to see that the solution (4) – (5) under the condition of instantaneous $(\tau_b = 0)$ setting of immobile plane to motion coincides with the Stokes solution [6, 8] – the second integral (5). But, as is obvious, in the absence of the second, immobile plane, which is at a finite distance from the moving one, the solution (5) will not have a linear distribution [16]. We also do not take into account the constant due to relation (2) (the finite power in the presence of dissipation cannot set the entire infinite space in motion at a constant speed). Information available on the Internet on laminar boundary layer research [7] indicates that all theories, starting with Blasius' work [18] on the flow around a flat plate, are nothing more than a good approximation of the parabolic law that corresponds to the motion of a fluid under the action of a longitudinal pressure gradient and, of course, does not correspond to the motion of a body in a fluid.

As mentioned above, the classical approach leads to unphysical results: after acceleration of the plane, with time (see (1)) the gradient of the velocity of the fluid in contact with the surface of the plane inexorably asymptotically tends to zero. It turns out that having accelerated the plane to a finite speed, we no longer need to make further efforts to maintain the motion at a constant speed. But, excuse us, where does viscous scattering go? Of course, it does not disappear. The boldness of these statements is confirmed by the results of work [7], where it is proved that in the boundary layer of an incompressible laminar fluid flow, the viscosity cannot remain constant in the absence of a longitudinal pressure gradient: it must be a function of the distance to the solid surface to ensure the constancy of the shear stress in the flow (see [7] for details). To avoid the physical inconsistency associated with the violation of the basic law of physics on the conservation of energy, we apply the ideas of a new approach initiated in [7].

3. Unsteady laminar non-gradient flow in the boundary layer: calculus of variation approach

Now it is appropriate to note that the approach used in [3] corresponds to the first ever calculus of variation principle of mechanics by Pierre's Maupertuis [30].

Unsteady non-gradient flow of incompressible fluid in the boundary layer is described by the generalized Navier-Stokes equation

$$\frac{\partial V_x}{\partial t}=\frac{\partial}{\partial y}\left(\mu\frac{\partial V_x}{\partial y}\right), \quad (6)$$

taking into account the variable viscosity coefficient $\mu(y,t)$ and also the following initial and two boundary conditions (hereafter all values have dimensionless form [7])

$$\begin{aligned} &V_x(t\to\infty)=1;\\ &V_x(y=0)=f(t);\\ &V_x(y\to\infty)\to 0. \end{aligned} \quad (7)$$

Therefore, the fluid flow functional can now be represented in the form

$$J=\int_0^\infty V_x\left(\frac{\partial V_x}{\partial t},\frac{\partial V_x}{\partial y}\right)dy\,. \quad (8)$$

The Euler equation of the extreme of the functional (8) has the following form

$$-\frac{\partial}{\partial t}\left(\frac{\frac{\partial V_x}{\partial V_x}}{\frac{\partial V_x}{\partial t}}\right)-\frac{\partial}{\partial y}\left(\frac{\frac{\partial V_x}{\partial V_x}}{\frac{\partial V_x}{\partial y}}\right)=0\,.$$

Since this equation must hold for any instant of time, from the asymptotic coincidence at large values of time (see also the gradient flow case below) we obtain

$$\frac{\partial}{\partial t}\left(\frac{\frac{\partial V_x}{\partial V_x}}{\frac{\partial V_x}{\partial t}}\right)=\frac{\partial}{\partial y}\left(\frac{\frac{\partial V_x}{\partial V_x}}{\frac{\partial V_x}{\partial y}}\right)=0\,.$$

As in [7], let's transform the previous equation into the form

$$\frac{\partial}{\partial t}\left(\frac{\partial V_x}{\partial t}\frac{1}{\frac{\partial^2 V_x}{\partial t^2}}\right)=\frac{\partial}{\partial y}\left(\frac{\partial V_x}{\partial y}\frac{1}{\frac{\partial^2 V_x}{\partial y^2}}\right)=0\,. \quad (9)$$

If we solve the problem directly, that is, using known approaches, something incomprehensible comes out. Let's try to use the method of variables separation by Fourier. According to this method,

$$V_x = V_x(t,y) = T(t)\cdot Y(y). \quad (10)$$

After substituting (10) into (9), (9) turns into

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$$\frac{d}{dt}\left(\frac{dT}{dt}\frac{1}{\frac{d^2T}{dt^2}}\right)=0,\quad \frac{d}{dy}\left(\frac{dY}{dy}\frac{1}{\frac{d^2Y}{dy^2}}\right)=0. \quad (11)$$

It follows from (11)

$$\frac{dT}{dt}=C_1^t\frac{d^2T}{dt^2},\quad \frac{dY}{dy}=C_1^y\frac{d^2Y}{dy^2} \quad (12)$$

with solutions in the form

$$T(t)=A_t+B_t\exp\left(\frac{t}{C_{1t}}\right),$$
$$Y(y)=A_y+B_y\exp\left(\frac{y}{C_{1y}}\right). \quad (13)$$

In (12), (13) $C_1^t, C_1^y, A_t, B_t, A_y, B_y$ are integration constants. Some of them are found from the following initial and boundary conditions

$$T(0)=0,\ T(\infty)=1;\ Y(0)=1,\ Y(\infty)=0. \quad (14)$$

As a result, the sought solution takes the form:

$$V_x(t,y)=\left(1-\exp(\frac{t}{C_1^t})\right)\exp(-y). \quad (15)$$

In the solution (15), the constant $C_1^y=-1$ is the same as for the case of steady flow [7]. The constant C_1^t will be defined later. It follows from (15) that

$$V_x(t,y)\to\exp(-y),\ t\to\infty,$$

and this is consistent with the results (flows with small Reynolds numbers) of modern works [31, 32] (see [3] for more details).

4. Unsteady laminar gradient flow in the boundary layer: calculus of variation approach

The motion is described by the generalized Navier-Stokes equation with variable viscosity inside the boundary layer

$$\rho\frac{\partial V_x}{\partial t}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial V_x}{\partial y}\right),$$

initial and boundary conditions (7). It also follows from the conditions of the problem that

$$-\frac{\partial p}{\partial x}=\text{Const}.$$

Further, taking into account the physics of the boundary layer (viscosity force of the same order as inertial force and pressure gradient), in dimensionless quantities $\text{Const}=1$. To use the calculus of variation approach, let us assume, as it was already done for the steady flow [7], that now

$$J=\int_0^\infty V_x\left(V_x,\frac{\partial V_x}{\partial t},\frac{\partial V_x}{\partial y}\right)dy. \quad (16)$$

The corresponding Euler equation for the extremum of the functional now has the following form

$$1-\frac{\partial}{\partial t}\left(\frac{\partial V_x}{\frac{\partial V_x}{\partial t}}\right)-\frac{\partial}{\partial y}\left(\frac{\partial V_x}{\frac{\partial V_x}{\partial y}}\right)=0. \quad (17)$$

We use the method of separation of variables again and obtain

$$\frac{d}{dt}\left(\frac{dT}{dt}\frac{1}{\frac{d^2T}{dt^2}}\right)=1-\frac{d}{dy}\left(\frac{dY}{dy}\frac{1}{\frac{d^2Y}{dy^2}}\right)=\text{Const}. \quad (18)$$

So far, we do not know the value of the constant in equations (18). Therefore, we will find their general solution. For this, as above, we find the first integrals (18). We have

$$\frac{dT}{dt}=\left(\text{Const}\cdot t+C_1^t\right)\frac{d^2T}{dt^2},$$
$$\frac{dY}{dy}=\left((1-\text{Const})\cdot y+C_1^y\right)\frac{d^2Y}{dy^2}. \quad (19)$$

The general solution corresponding to (19) is

$$V_x(y,t)=\left[A_y+B_y\left(y-\frac{C_1^y}{-1+\text{Const}}\right)^{\frac{-2+\text{Const}}{-1+\text{Const}}}\right]\times$$

$$\times\left[A_t+B_t\left(t+\frac{C_1^t}{Const}\right)^{\frac{1+Const}{Const}}\right]. \qquad (20)$$

The solution (20) asymptotically, when $Const\to 0$, turns into a solution of the steady problem:

$$\lim_{Const\to 0} V_x(y,t)=\left[A_y+B_y\left(y+C_1^y\right)^2\right]\times$$
$$\times\left[A_t+B_t\exp\left(\frac{t}{C_1^t}\right)\right]. \qquad (21)$$

So, let's set $Const=0$. After meeting all the boundary conditions (by spatial coordinate and by time), we get the following solution

$$V_X(y,t)=\left(1-\exp\left(\frac{t}{C_1^t}\right)\right)\cdot y(2-y). \qquad (22)$$

As one can see, at $C_{1t}<0$

$$V_X(y,t)\to y(2-y),\ t\to\infty. \qquad (23)$$

Expression (23) completely coincides with the solution of the problem in the case of steady flow [7].

5. Functions of viscosity, shear stress and power of friction force

Unlike steady motion, as follows from the solutions obtained above, now the viscosity is a function of time and spatial coordinate (in the case of a non-gradient boundary layer):

$$\mu=\mu(t,y). \qquad (24)$$

For a non-gradient boundary layer, substituting solution (15) into equation (5) leads to the relation:

$$-\frac{1}{C_1^t}\cdot\exp\left(\frac{t}{C_1^t}\right)\exp(-y)+\varphi(t)=$$
$$=\mu(t,y)\left[1-\exp\left(\frac{t}{C_1^t}\right)\right]\exp(-y),$$

from which it follows that

$$\mu(t,y)=\frac{\exp\left(\frac{t}{C_1^t}\right)\exp(-y)+\varphi(t)}{C_1^t\left[1-\exp\left(\frac{t}{C_1^t}\right)\right]\exp(-y)}, \qquad (25)$$

with $\varphi(t)$ to be some function of time arising from partial integration. Since the steady flow can be considered as the limiting case of the non-steady one for $t\to\infty$, then, comparing with the solution for steady problem, we obtain that for any $t>0$

$$\mu(t,y)=\frac{-\frac{1}{C_1^t}\cdot\exp\left(\frac{t}{C_1^t}\right)\exp(-y)+1}{\left[1-\exp\left(\frac{t}{C_1^t}\right)\right]\exp(-y)}\to\exp(y),\ t\to\infty.$$

For a gradient flow, according to solution (22) and equation (5), we obtain

$$-\frac{1}{C_1^t}\exp\left(\frac{t}{C_1^t}\right)y(2-y)=$$
$$=1+\frac{\partial}{\partial y}\left(\mu(2-2y)\left(1-\exp\left(\frac{t}{C_1^t}\right)\right)\right).$$

Whence, after integration over y, we obtain:

$$\mu(t,y)=\frac{\psi(t)-\frac{1}{C_1^t}\cdot\exp\left(\frac{t}{C_1^t}\right)\cdot\left(y^2-\frac{y^3}{3}\right)-y}{2(1-y)\left(1-\exp\left(\frac{t}{C_1^t}\right)\right)}. \qquad (26)$$

For $t\to\infty$ the flow becomes stationary. Then, from the condition of agreement with the steady flow, we obtain $\psi(t)=1$. Finally, the viscosity function has the following form

$$\mu(t,y)=\frac{1-y-\frac{1}{C_1^t}\cdot\exp\left(\frac{t}{C_1^t}\right)\cdot\left(y^2-\frac{y^3}{3}\right)}{2(1-y)\left(1-\exp\left(\frac{t}{C_1^t}\right)\right)}\to\frac{1}{2},\ t\to\infty.$$

The viscous stress functions are found by the formula

$$\tau_{xy}=\mu\frac{\partial V_X}{\partial y}. \qquad (27)$$

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For a non-gradient boundary layer, according to (25) and (27), we obtain

$$\tau_{xy} = -\left(1-\frac{1}{C_1^t}\cdot\exp\left(\frac{t}{C_1^t}\right)\exp(-y)\right)\to -1, \quad t\to\infty. \qquad (28)$$

For the case of a gradient boundary layer, it is obtained similarly from (26) and (27),

$$\tau_{xy} = 1-y-\frac{1}{C_1^t}\exp\left(\frac{t}{C_1^t}\right)\left(y^2-\frac{y^3}{3}\right)\to 1-y, \quad t\to\infty. \qquad (29)$$

It is important to note that when a fluid flows along an immobile body (gradient boundary layer), the surface shear stress acting from the side of the fluid on the plane

$$\tau_{xy}=1,\ y=0,\ t\geq 0$$

always constant and, of course, directed along the flow of the fluid, since it is the motion of the fluid that causes the appearance of stresses.

In practice, it is important to know at each moment of time the power of the frictional force, which is determined by the formula

$$P=\tau_{xy}\cdot V_x. \qquad (30)$$

According to (30), for a gradient-free boundary layer, taking into account the expressions for velocity (15) and shear stress (28), we obtain

$$P(t,y)=\left(1-\exp\left(\frac{t}{C_1^t}\right)\right)\exp(-y)\times \times\left(-1+\frac{1}{C_1^t}\times\exp\left(\frac{t}{C_1^t}\right)\exp(-y)\right). \qquad (31)$$

For the gradient boundary layer, after substituting expressions (22), (29) into (30), we obtain

$$P(t,y)=\left(1-\exp\left(\frac{t}{C_1^t}\right)\right)\cdot y(2-y)\times \times\left(1-y-\frac{1}{C_1^t}\exp\left(\frac{t}{C_1^t}\right)\left(y^2-\frac{y^3}{3}\right)\right). \qquad (32)$$

If we compare expressions (28) – (30) with (1), the meaning of the above considerations will become clear: relation (24) is the cornerstone of this entire theory. Due to the possibility of changing the molecular viscosity through the boundary layer, physically appropriate solutions of non-steady problems are obtained. These solutions, on the one hand, do not contradict the law of conservation of energy (the tangential stresses on the surface of the plane do not disappear during the transition from non-steady to steady motion), and on the other hand, they are completely asymptotically consistent with their analogues for steady problems. Finally, these analogues, most importantly, are consistent with the relevant results of existing experiments and theories [7].

To obtain explicit graphical dependencies, it is also necessary to determine the constant C_1^t in solutions (15), (22). Here it is appropriate to use the recent work of Schreas Mandre [33], where a calculus of variation problem on the method of acceleration of a flat plate of finite length with a limitation on the available power is considered. For the acceleration function, the following relation was obtained there ([33], formula (3.27a))

$$f(t')=\left(1-\exp(-2.62t')\right)^{1/4}. \qquad (33)$$

If we try to find C_1^t in the solution (15) for $y=0$ from the condition of equality to (33), then we obtain

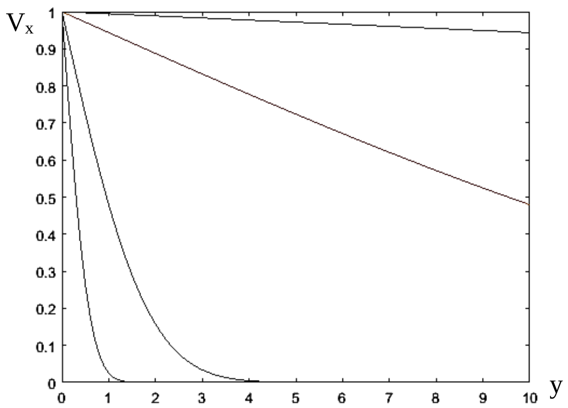
$$C_1^t \approx -0.25.$$

However, the analysis of graphic data, as well as the absolute analysis of the tendency of the velocity to zero (further decline does not exceed 1%), indicates in favor of the fact that

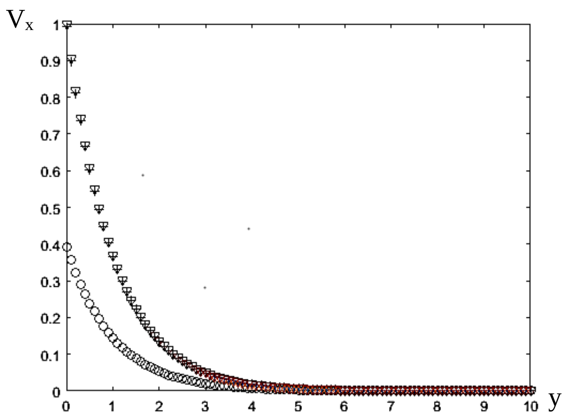
$$C_1^t \approx -0.2. \qquad (34)$$

Figure 1 shows the time evolution of the velocity distribution. It is clearly visible that the obtained solution (15) reaches an asymptote (see Fig. 1, b), that is, a steady solution, which is consistent with the exponential decrease of the amplitude (see [6, 7]). Moreover, immediately after acceleration (dimensionless time is equal to one), the curves practically coincide at the following moments of time. What cannot be said about the self-similar solution: over time, this solution approaches a constant value in physical coordinates (see Fig. 1, a). And this cannot be achieved due to the presence of viscous dissipation. If we consider the uppermost curve in Fig. 1, a as an asymptote for a steady flow, we will not find experimental data on such a velocity distributions [16].

Figure 2 shows a comparison of shear stress functions on the surface of the moving plane. These are the solution (28) and the Rayleigh solution (see [8], (17), (18)). Since the Navier-Stokes equations are equations of stress dynamics, the focus is on the stress function, and especially on the surface. According to Rayleigh's solution, the shear stress only increases during acceleration and then, for unknown reasons, decreases to zero, allowing, as mentioned above, the existence of a perpetual motion without further external energy input. As for the time dependence of the shear stress in the obtained solution, this dependence indicates a constant value, which corresponds to a steady (constant – in this problem) value, which is consistent with the existing ideas about this type of motion. The fact that the shear stress at the initial moment of time is maximum is fully consistent with such a concept as friction of rest: this phenomenon occurs precisely during the imparting of momentum to the body, in fact, accelerating it to a constant speed. From a technical, as well as an energy point of view, the time dependence of the power spent on acceleration, and then on maintaining steady motion, is of interest.



a



b

Fig. 1. Time evolution of the Stokes solution (Fig. a)) and solution (15) (Fig. b). In fig. and moments of dimensionless time 0.1.1 00.10000 are given; in fig. b are equal to 0.1, 1 and 2, respectively

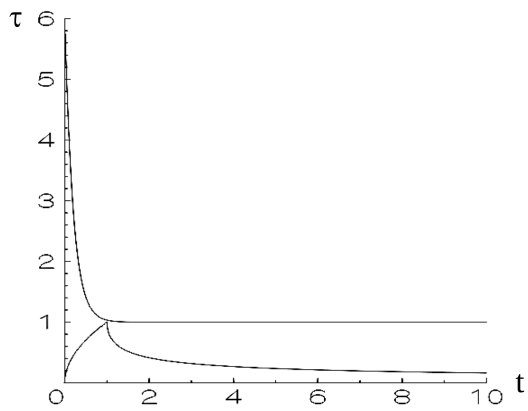


Fig. 2. Dependence of shear stress on time on a flat surface (dimensionless values): upper curve – solution (28), lower one is Rayleigh solution, [4], formulas (17), (18)

Figure 3 shows the dependences for the power of the friction force on the surface of the moving plane. It is clearly visible that the power required for acceleration of the plane increases both during acceleration with constant acceleration (Rayleigh [8]) and according to relations (15) and (28). However, if the specified growth for the model presented in this article is replaced by a constant value that is reached (asymptotes), then, according to Rayleigh's solution, the power required to maintain steady motion decreases in time to zero.

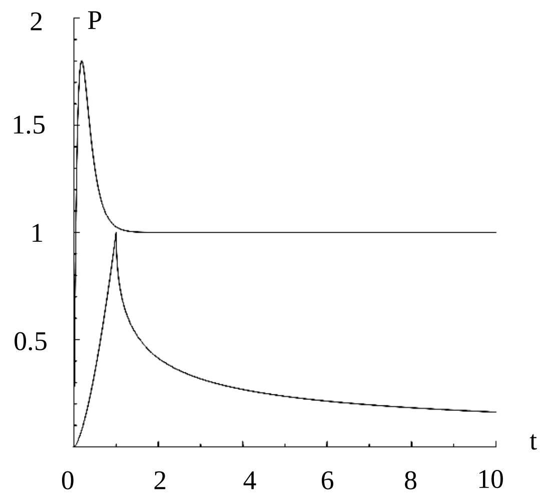


Fig. 3. Dependence of the power of the force of friction against a flat surface on time (dimensionless values): the upper curve is the solution (31), the lower one is the Rayleigh solution [4], formulas (17), (18)

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Discussion

Of course, it is not entirely correct to use the results of the problem of the motion of an infinite plane for bodies with finite dimensions, at least in the direction of motion. But, you see, viscous dissipation does not disappear anywhere even in the case of an infinite region (formula (2)), and therefore, at each moment of time after reaching a steady mode of motion, a finite power is needed to maintain this motion. On the other hand, it is known that when calculating friction, both for an external problem and for an internal one, resistance exists on the entire surface of a solid body. For example, when calculating the resistance of the pipeline, the length is of significant importance: the longer the pipe, the more powerful the pump is needed to pump the liquid. There is no such phenomenon when the resistance does not increase after a certain region of the pipe. Such a conclusion, if it is assumed as a consequence of the constancy of molecular viscosity, contradicts reality.

In addition, if the plane is semi-infinite, then when flowing along it, there is a region of establishment of the current, beyond which the same motion as for an infinite plane takes place. The existing modern theory has many shortcomings, which are gaps in our knowledge. In order to eliminate these gaps, as it turns out, it is necessary to develop new approaches to setting and solving mathematical problems – also new ones. As for the approach presented in this work, it has proven itself well in the problem of steady flow [7]. In particular, the results obtained in [7] agree well with the experimental data.

Conclusions

As shown in this paper, at speeds not exceeding the order of the Mach number Ma = 0.2 (that is, up to 70 m/s), which is characteristic of the take-off mode and from a mathematical point of view corresponds to an incompressible flow, in the approximation of the laminar boundary layer, the molecular viscosity is variable depending on the distance to the body surface and time. The currently used Stokes model, based on the constancy of molecular viscosity for an incompressible flow, leads to deliberately erroneous results: after the acceleration of a body in a viscous fluid over time, the possibility of the existence of a perpetual motion is revealed.

In steady flow around an immobile body with a fluid flow uniform at infinity, the molecular viscosity can be considered constant, and for the simplest geometry, which is an infinite plane, the boundary layer is described by a parabolic law for the distribution of velocity. What cannot be said about the motion of a body in a still fluid. Here, on the example of an infinite plane, the condition of constancy of shear stress across the boundary layer (due to the absence of a longitudinal pressure gradient) inexorably leads to the requirement of variable character of molecular viscosity. The approach outlined in this article made it possible to obtain a physically consistent description of the boundary layer of an incompressible laminar flow, which is expressed in the presence at any time of the frictional stress of a moving body against a still fluid – or vice versa.

Finally, in order to answer the question of the practical use of the above results, we point out the need to rethink the conduct of experiments in wind tunnels. Although the main component of the lift force associated with the redirection of the air flow by the wing remains unchanged, in the conditions of the wind tunnel it is not possible to obtain the structure of the non-gradient boundary layer, the same as in the conditions of flight. Therefore, as a recommendation, we suggest rethinking the very technology of the experiment and think about how to create the motion of the test sample in laboratory conditions and thus bring the experiment as close as possible to a real flight.

As a further study, it is possible to consider the variations of the change in acceleration during acceleration of the aircraft and its effect on the characteristics of the set motion.

Contribution of authors: conceptualization – **Pavlo Lukianov**, **Lin Song**; formulation of task – **Pavlo Lukianov**; analysis – **Pavlo Lukianov**, **Lin Song**; software – **Lin Song**; development of mathematical model – **Pavlo Lukianov**; analysis of results – **Pavlo Lukianov**, **Lin Song**.

All the authors have read and agreed to the published version of the manuscript.

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НЕСТАЦІОНАРНИЙ НЕСТИСЛИВИЙ ЛАМІНАРНИЙ ПРИМЕЖОВИЙ ШАР: ЗМІННА У ЧАСІ ТА ПРОСТОРІ МОЛЕКУЛЯРНА В’ЯЗКІСТЬ

Павло Лук’янов, Лінь Сун

Предметом даної роботи є два підходи до опису ламінарної нестаціонарної течії нестисливої рідини в примежового шарі. У першому підході в'язкість рідини і прискорення, з яким приводиться в рух площина, вважають сталими. По суті, це задача Релея. Розв'язок, отриманий на основі цих припущень, асимптотично збігається до відомого автомодельного розв'язку Стокса. Важливо, що розв’язки Стокса і Релея асимптотично при великих значеннях часу відповідають зникненню напружень зсуву між рідиною і рухомою площиною після прискорення. Виходить парадокс: виведені Стоксом рівняння для опису внутрішнього тертя свідчать про відсутність того самого тертя між рухомим тілом і рідиною. Оскільки при дослідженні методами варіаційного числення виявилося, що всередині стаціонарного примежового шару молекулярна в'язкість повинна залежати від відстані до рухомої поверхні, була розглянута відповідна нестаціонарна задача. У результаті, як і раніше для стаціонарного випадку, отримані розв’язки, що описують як безградієнтні, так і градієнтні течії нестисливої рідини в примежовому шарі. Асимптотичний аналіз переходу до стаціонарної течії свідчить про узгодженість цих розв’язків. Для випадку безградієнтної течії проведено порівняння класичного розв’язку з розв’язком, що відповідає екстремуму втрати рідини, що переноситься рухомою поверхнею. Показано, що згідно з розв’язком, отриманим на основі варіаційного підходу, напруження зсуву на поверхні після встановлення руху нікуди не зникає, а, як і очікувалося, набуває сталого значення. **Методи дослідження** є суто теоретичними, а результати аналізуються шляхом порівняння з наявними теоретичними та експериментальними даними та відповідністю до фундаментальних законів фізики, зокрема закону збереження енергії. Ці методи базуються на побудові аналітичних математичних моделей, що представляють собою диференціальні рівняння в частинних похідних, доповнених відповідними фізичними початковими та граничними умовами. Крім того, використовуються диференціальні рівняння Ейлера теорії екстремуму функціонала (в даній роботі це екстремум втрати рідини поперек перерізу примежового шару). При розв'язуванні цих рівнянь використовується відомий метод розділення змінних Фур'є. Довільні функції часу, що виникають при частинному інтегруванні (за однією зі змінних – просторовою координатою), визначаються з умов асимптотичної прямування розв’язків нестаціонарних задач до відповідних до них розв’язків стаціонарних задач. **Висновки.** Представлені результати мають принципове значення для розуміння фізики обтікання частин літака, оскільки вказують на суперечливість існуючого уявлення про оборотність прямої та оберненої задач: руху тіла в нерухомій рідині та обтікання рідиною нерухомого тіла.

Ключові слова: літак; ламінарний примежовий шар; нестаціонарна нестислива течія; змінна молекулярна в'язкість.

Павло Лук’янов – канд. фіз.-мат. наук, старш. наук. співроб., доц. каф. гідрогазових систем, Національний авіаційний університет, Київ, Україна.

Лінь Сун – магістр, асп. каф. гідрогазових систем, Національний авіаційний університет, Київ, Україна.

Pavlo Lukianov – Candidate of Physics and Mathematics Sciences, Senior Researcher, Associate Professor of the Department the Hydro-Gas Systems, National Aviation University, Kyiv, Ukraine,
e-mail: Pavlo.Lukianov@npp.nau.edu.ua, ORCID: 0000-0002-5043-6182.

Lin Song – Master of Technical Sciences, PhD Student of the Department of Hydro-Gas Systems, National Aviation University, Kyiv, Ukraine,
e-mail: 592123500@qq.com, ORCID: 0000-0001-6328-4889.