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UNSTEADY FLOW OF BUBBLE LIQUID IN HYDRAULIC SYSTEMS OF AIRCRAFT AND HELICOPTERS

The subject of this work *is the phenomenon of a water hammer in a liquid that contains a small volume of gas bubbles. Historically, this phenomenon began to be studied as the dynamics of gas bubbles (Rayleigh-Pleset equation). Today, thanks to progress in computer technology, this phenomenon is studied at the level of bubble deformation during hydraulic shock. Another approach is to consider the dynamics of a multiphase (two-phase) medium in the form of a bubbly liquid. After several assumptions, the main one being a relatively small gas content in the liquid, the model consists of two differential equations with respect to the shock wave propagation speed and the resulting pressure perturbations. The specified system of equations differs from the corresponding classical water hammer equations: they consider the convection of the velocity field. In addition, the friction of the liquid against the wall according to the Weisbach-Darcy model is considered. Because of the small content of gas bubbles, the Weissbach-Darcy friction is approximated in the same way as in a homogeneous liquid, i.e., in a certain sense, greater than the real friction. Maybe that is why more or less physical results are obtained only for small values of the dimensionless parameter responsible for the friction of the liquid against the wall. It concerns the non-contradiction of the assumptions and the results obtained on their basis. Thus, in the front region of the shock pulse, where the pressure increases, the radial velocity of the bubbles is negative; however, for relatively large values of the friction parameter, the maximum pressure disturbance moves from the center of the shock pulse. This contradicts the assumption about compression: after passing the maximum pressure, gas bubbles expand due to a decrease in pressure. The graphical dependence obtained in this study are compared with the results related to a homogeneous liquid. They agree, but the shock pulse in a bubbly liquid is not as concentrated in space as that in a homogeneous liquid. Its length is 10-12 times greater than the corresponding value in a homogeneous liquid.* ***Research methods*** *are purely theoretical. The well-known bubble liquid model is used as a single-speed model continuum. Differential equations are solved analytically, approximately (series expansion), and numerically. In addition, the original approach of obtaining an analytical solution of an autonomous system is used-finding the function of pressure disturbances from the velocity of propagation of the shock pulse (and vice versa).* ***Conclusions.*** *A simple one-dimensional hydraulic model of shock wave (impulse) propagation in a bubbly liquid is proposed. In contrast to classical ideas (solutions) about a water hammer, which consists of two waves of opposite directions of propagation, a shock pulse is a region of pressure disturbances in which the speed of motion of fluid particles is also variable – from the maximum value to almost zero.*

Keywords: *aircraft; helicopter; structural element; hydraulic shock; two-phase flow; stress; surface deformation; fatigue.*

Introduction

The motion of liquid in the hydraulic system of airplanes and helicopters is accompanied by the phenomenon of cavitation, which leads to the appearance of gas bubbles in the liquid. Since the seals are not perfect, air gets into the region of almost zero absolute pressure. This air dissolves inside the droplet liquid. The presence of gas in the liquid, in turn, is dangerous, as it leads to the occurrence of water hammer in the system. Therefore, the phenomenon of the propagation of a shock pulse (water hummer) in a bubbly liquid is of both scientific and practical interest.

When calculating the propagation, reflection and other processes related to the shock wave, it is very important to have as accurate information as possible about the initial distribution of hydrodynamic characteristics in order to further use this information in the numerical calculations of surface deformation and its possible fatigue. The fact is that the physical nature of shock pulse propagation is significantly non-linear and this enables the existence of different types of motion. Therefore, in order to study the desired type of motion (mode), the initial profile of the shock pulse should be set correctly (appropriately) - the corresponding fields of pressure and propagation velocity disturbances.

The main feature of shock wave propagation in a two-phase medium consisting of liquid and gas is that the volume mass is concentrated in the liquid phase, while the compressibility of the medium is completely determined by the compressibility of the gas in the bubbles [1]. The cited paper has important experimental data for comparison. Thus, in the second chapter [1] in Fig. 2 there is a close to square parabolic dependence of the dimensionless velocity (Mach number) on the normalized pressure value. Approximately such a dependence as a part of the exponential function is also obtained in this work (see Fig. 1).

The work [2] is devoted to the equations proposed in [3], and more precisely to how these equations can be obtained from microscopic equations in a special analytical limit, which is considered in detail. Among others, the mathematical model uses differential equations for the growth of the bubble radius. The task at the microscopic level is formulated in the cited work on the basis of a number of assumptions about the physical characteristics of fluid motion.

1. The center of the bubble does not move.

2. Bubbles have a spherical shape with a uniform distribution of internal pressure.

3. Surface pressure, viscosity, or thermal conductivity are not explicitly included.

4. The liquid is almost compressible, has a constant density and speed of sound, and the flow itself is vortex-free, although in modern papers the formation of a vortex structure in a bubble is observed.

The work [4] is a continuation of studies in [2]. As the authors of the paper [4] note in the introduction, they are interested in the physical refinement of the model, its features, such as interphase friction or mutual interaction of bubbles. At the same time, the main interest is to investigate how far the model mathematically corresponds to the first order of the volume concentration of the gas and can explain the observed behavior of bubbly liquids.

A laboratory and numerical experiment to study the propagation of a shock wave in a liquid containing bubbles is presented in [5]. The reason for writing the cited work was that the theoretical and experimental results were very different. The reason for these discrepancies, as it turned out, is the significant influence of the spatial distribution of bubbles on the structure of the shock wave. The cited work indicates that the theoretical results correspond to the uniform distribution of bubbles in the liquid volume. While in many experiments this condition was not met: the bubbles filled the volume in a chaotic manner, far from being homogeneous.

More specific studies on the transformation of the momentum into a shock wave to a bubbly liquid are presented in [6]. The photographs presented in the cited work indicate a fairly uniform distribution of bubbles in the liquid volume, which indicates the possibility of comparison with the theoretical results of this work.

The work [7] is also devoted to the study of shock wave propagation in bubbly liquids. This is how the ideas of articles [2,5] are developed in the cited work. But the authors went further - they already took into account the heat and mass exchange between the liquid and the bubble. The energy equation for the gas inside the bubble is solved analytically. The results of the numerical experiment on the attenuation of oscillations behind the shock wave front obtained in the paper are in good agreement with [5].

A relatively early paper [1] contains a comprehensive physical description of the process of the shock wave propagation in a liquid with gas bubbles. The main focus of research in this papaer is the physical analysis of shock wave saturation depending on the volume concentration of bubbles.

Experimental studies of the dynamics and structure of pressure waves of moderate intensity in a liquid with gas bubbles of one or two sizes in a wide range of waves, as well as studies of the behavior of a gas bubble during the passage of a wave - all this is presented in the paper [8].

The passage of a shock wave through a liquid with a significant (10%) volume content of gas bubbles was studied numerically in [9]. Special attention is paid there to bubble interaction and bubble deformation.

Attempts to analytically describe the shock wave in a liquid with gas bubbles include the work [10]. In it, in particular, the passage of a shock wave is modeled using the Kordeweg-de-Fries-Burgers equation. An interesting point is the introduction of effective viscosity. If we combine modern data on molecular viscosity in the boundary layer, it becomes clear that the viscosity will really depend on the size of the bubbles.

Another one of the first theoretical works on the study of shock waves in liquids with bubbles is the paper [3]. The studies in the cited paper are mainly based on equations describing the radius of the bubble. In this article, there is a reference to the report [11], in which the ratio of conservation of mass and the momentum across the shock wave is established.

The quasi-homogeneous model of Zvik [12] was used to study long-wave disturbances in a gas-liquid mixture with a small volumetric gas content in work [13].

The work [14] is devoted to the experimental study of the formation of a shock wave by increasing the steepness of compression waves.

1. Problem formulation

On the basis of a nonlinear model of unsteady flow in a liquid containing a small amount of gas bubbles, formulate and solve the problem of water hammer, in particular, find analytical and numerical solutions to this problem.

Study the effect of bubble fraction on the process of propagation of the shock pulse.

Carry out a comparative analysis of the obtained solution with a similar solution in a homogeneous liquid.

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2. Unsteady flow of a drop liquid in the presence of a small fraction of gas bubbles

Further research is based on such assumptions [15]:
- weak disturbances;
- homogeneous monodisperse mixture;
- the liquid is not compressible;
- a single-speed scheme with a polytropic gas and an effective viscosity is used for calculations.

The system of differential equations consists of [15]:

– momentum conservation equation

$$\rho\left(\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}\right)+\frac{\partial p}{\partial x}=0, \quad \rho\approx\rho_1^0\alpha_1; \tag{1}$$

– mass conservation equation

$$\left(\frac{\partial\rho_1^0}{\partial t}+V\frac{\partial\rho_1^0}{\partial x}+\rho_1^0\frac{\partial V}{\partial x}\right)=\frac{3\alpha_2 w_{1a}}{\alpha_1 a}\rho_1^0; \tag{2}$$

– equation of the acoustic compressibility of the carrier fluid:

$$\rho_1^0=\rho_{10}^0+\frac{p-p_0}{c_1^2}. \tag{3}$$

In equations (1) - (3) there are six parameters: α_1, α_2 – concentration of phases, a, w_{1a}, c_1 – radius of the bubble, radial velocity in the bubble, speed of sound in the liquid, ρ_{10}^0 – undisturbed density of the main phase (liquid). More details are in [15], equation (6.2.1). Although these equations are specified in [15] as corresponding to small perturbations (see first assumption above), it follows from equation (3) only that

$$\frac{p-p_0}{c_1^2}<<\rho_{10}^0. \tag{4}$$

Therefore, inequality (4) is valid for typical values $\rho_{10}^0=O(10^3)$ and $c_1\approx1.5\cdot10^3$ gives $p-p_0\approx2\cdot10^9$ ≈ 20000 atm. Disturbances in pressure during a shock wave are of the order of one hundred to two hundred of atmosphere pressure and they correspond to "small" ones relatively to 20 000 atm. (within 1-2%).

Substitute the second equation (1) into the first, and equation (3) into equation (2), we obtain:

$$\alpha_1\rho_1^0\left(\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}\right)+\frac{\partial p}{\partial x}=0, \tag{5}$$

$$\frac{1}{c_1^2}\left(\frac{\partial p}{\partial t}+V\frac{\partial p}{\partial x}\right)+\left(\rho_{10}^0+\frac{p-p_0}{c_1^2}\right)\frac{\partial V}{\partial x}= =\frac{3\alpha_2 w_{1a}}{\alpha_1 a}\left(\rho_{10}^0+\frac{p-p_0}{c_1^2}\right). \tag{6}$$

It should be noted that from (5), (6) in the case of a homogeneous liquid, we have [16, 17]:

$$\rho\left(\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}\right)+\frac{\partial p}{\partial x}=0,$$

$$\frac{1}{c_1^2}\frac{\partial p}{\partial t}+\rho\frac{\partial V}{\partial x}=0.$$

This is important because it is easy to make a mistake and leave the term corresponding to the convection of the pressure field, but it appeared only due to the inhomogeneity of the fluid and, therefore, has no relevance when considering shock wave in a homogeneous fluid.

Hereafter, it is more convenient to consider not the pressure itself, but its disturbance $p'=p-p_0$. At the same time, the following ratio is valid:

$$\frac{\partial p}{\partial x}=\frac{\partial(p-p_0)}{\partial x}=\frac{\partial p'}{\partial x}. \tag{7}$$

The system of equations (5), (6), taking into account (7), transforms to such the form:

$$\alpha_1\left(\rho_{10}^o+\frac{p'}{c_1^2}\right)\left(\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}\right)+\frac{\partial p'}{\partial x}=0, \tag{8}$$

$$\left(\rho_{10}^o+\frac{p'}{c_1^2}\right)\left(\frac{\partial V}{\partial x}-3\frac{\alpha_2}{\alpha_1}\frac{w_{1a}}{a}\right)+\frac{1}{c_1^2}\left(V\frac{\partial p'}{\partial x}+\frac{\partial p'}{\partial t}\right)=0. \tag{9}$$

3. Derivation of dimensionless equations in self-similar variables

Let's make the system of equations (8), (9) dimensionless. This leads to simplification of the mathematical formulation of the problem.

Introduce the scales of length, speed, time and density as follows:

$$[x]=L,\ [V]=c_1,\ [t]=\frac{[x]}{[V]}=\frac{L}{c_1},\ [\rho]=\rho_{10}^o.$$

Substituting these ratios into equation (1), we obtain the following relation:

$$\rho_{10}^{o}\alpha_1\left(1+\frac{\bar{p}'}{c_1^2}\right)\frac{c_1^2}{L}\left(\frac{\partial\bar{V}}{\partial\bar{t}}+\bar{V}\frac{\partial\bar{V}}{\partial\bar{x}}\right)+\frac{\partial p'}{\partial x}\frac{[p']}{L}=0. \quad (10)$$

It follows from equation (10) that

$$[p']=\rho_{10}^{0}\alpha_1 c_1^2. \quad (11)$$

$$(1+\alpha_1\bar{p}')\left(\frac{\partial\bar{V}}{\partial\bar{t}}+\bar{V}\frac{\partial\bar{V}}{\partial\bar{x}}\right)+\frac{\partial p'}{\partial x}=0. \quad (12)$$

Equation (12), however, does not contain viscous effects. This applies primarily to wall friction, which can be described by the Weissbach-Darcy model [18, 19]. Taking into account the scales just introduced, the corresponding term in the equation for the conservation of momentum has the following form:

$$\frac{\lambda}{4R_H}\rho_{10}^{0}(1+\alpha_1\bar{p}')c_1^2\bar{V}\left|\bar{V}\right|. \quad (13)$$

In (13) R_H is the hydraulic radius.

Since when dimensioning (8) we divided by $\rho_{10}^{o}\alpha_1 c_1^2/L$, we will perform the same procedure with expression (13). As a result, we get another new dimensionless parameter

$$\bar{\lambda}=\frac{\lambda L}{4R_H}. \quad (14)$$

Taking into account the effect of friction against the wall, now equation (12) turns into the following:

$$(1+\alpha_1\bar{p}')\left(\frac{\partial\bar{V}}{\partial\bar{t}}+\bar{V}\frac{\partial\bar{V}}{\partial\bar{x}}\right)+\frac{\partial\bar{p}'}{\partial\bar{x}}+\bar{\lambda}\,(1+\bar{p}')\bar{V}\left|\bar{V}\right|=0. \quad (15)$$

Let's highlight the scales of all motions in equation (9) and rewrite it in the following form:

$$\rho_{10}^{o}(1+\bar{p}')\frac{c_1}{L}\left(\frac{\partial\bar{V}}{\partial\bar{x}}-3\frac{\alpha_2}{\alpha_1 a}\frac{w_{1a}}{c_1}L\right)+$$

$$+\frac{1}{c_1^2}c_1\rho_{10}^{o}\frac{c_1^2}{L}\bar{V}\frac{\partial\bar{p}'}{\partial\bar{x}}+\frac{1}{c_1^2}\rho_{10}^{o}\frac{c_1^2}{L}\frac{\partial\bar{p}'}{\partial\bar{t}}=0.$$

After simplifications, we obtain:

$$(1+\bar{p}')\left(\frac{\partial\bar{V}}{\partial\bar{x}}-3\frac{\alpha_2}{\alpha_1}\frac{w_{1a}}{c_1}\frac{L}{a}\right)+\left(\bar{V}\frac{\partial\bar{p}'}{\partial\bar{x}}+\frac{\partial\bar{p}'}{\partial\bar{t}}\right)=0.$$

We will also introduce a new parameter

$$Bb=3\frac{\alpha_2}{\alpha_1}\frac{w_{1a}}{c_1}\frac{L}{a}. \quad (16)$$

The parameter Bb in expression (16) is obviously responsible for the influence of bubbles on the flow dynamics. Therefore, the second dimensionless equation (conservation of mass), taking into account (7), will be:

$$(1+\bar{p}')\left(\frac{\partial\bar{V}}{\partial\bar{x}}-Bb\right)+\left(\bar{V}\frac{\partial\bar{p}'}{\partial\bar{x}}+\frac{\partial\bar{p}'}{\partial\bar{t}}\right)=0. \quad (17)$$

The system of equations (15), (17) is the desired one. It contains three dimensionless parameters: α_1, $\bar{\lambda}$, Bb. At the same time, two of them refer to the content of bubbles in the liquid, and the third corresponds to taking into account the friction against the wall. It is also assumed that the friction against the wall is approximately the same as without the content of bubbles, since this parameter is small (several percent).

The system of equations (6), (8) should be supplemented with initial and, if there are boundaries, boundary conditions. Therefore, as in previous studies [16, 17], to simplify the mathematical problem and formulate the specified conditions, it is convenient to come to the self-similar variable. Recall that c_1 is the speed of sound in the carrier phase, that is, the liquid. Let's change (x,t) for self-similar variable

$$\eta(x,t)=x-c_1 t,$$

or in dimensionless quantities

$$\bar{\eta}(\bar{x},\bar{t})=\bar{x}-\bar{t}. \quad (18)$$

In self-similar variables, the system of equations (15), (17) takes the following form:

$$(1+\alpha_1\bar{p}')\left(-\frac{d\bar{V}}{d\bar{\eta}}+\bar{V}\frac{\partial\bar{V}}{d\bar{\eta}}\right)+\frac{d\bar{p}'}{d\bar{\eta}}+\bar{\lambda}\,(1+\bar{p}')\bar{V}\left|\bar{V}\right|=0, \quad (19)$$

$$(1+\bar{p}')\left(\frac{d\bar{V}}{d\bar{\eta}}-Bb\right)+\left(\bar{V}\frac{d\bar{p}'}{d\bar{\eta}}-\frac{d\bar{p}'}{d\bar{\eta}}\right)=0. \quad (20)$$

4. Solution of the problem

4.1. Propagation of a shock wave in a bubbly liquid without taking into account the friction against the walls

For a clear understanding of the influence of various physical factors on the process of shock wave propagation, let's start with a model where friction against the wall is not taken into account. We immediately rewrite the system of equations (19), (20), without taking into account the Weisbach-Darcy friction relatively unknowns $d\bar{p}'/d\bar{\eta}$, $d\bar{V}'/d\bar{\eta}$

$$\left(1+\alpha_1\bar{p}'\right)\left(\bar{V}-1\right)\frac{d\bar{V}}{d\bar{\eta}}+\frac{d\bar{p}'}{d\bar{\eta}}=0, \qquad (21)$$

$$\left(1+\alpha_1\bar{p}'\right)\left(\bar{V}-1\right)\frac{d\bar{V}}{d\bar{\eta}}+\frac{d\bar{p}'}{d\bar{\eta}}=0, \qquad (22)$$

We use Kramer's method. The system matrix (21), (22) has the form:

$$\left(\begin{array}{cc|c}\left(1+\alpha_1\bar{p}'\right)\left(\bar{V}\text{-}1\right) & 1 & 0\\ \left(1+\bar{p}'\right) & \left(\bar{V}\text{-}1\right) & Bb\left(1+\bar{p}'\right)\end{array}\right). \qquad (23)$$

The determinant of matrix (23) is equal to:

$$\Delta=\left(1+\alpha_1\bar{p}'\right)\left(\bar{V}\text{-}1\right)^2-\left(1+\bar{p}'\right).$$

There are two other determinants

$$\Delta_1=\begin{vmatrix}0 & 1\\ Bb\left(1+\bar{p}'\right) & \left(\bar{V}\text{-}1\right)\end{vmatrix}=-Bb\left(1+\bar{p}'\right),$$

$$\Delta_2=\begin{vmatrix}\left(1+\alpha_1\bar{p}'\right)\left(\bar{V}\text{-}1\right) & 0\\ \left(1+\bar{p}'\right) & Bb\left(1+\bar{p}'\right)\end{vmatrix}=Bb\left(1+\alpha_1\bar{p}'\right)\left(1+\bar{p}'\right)\left(\bar{V}\text{-}1\right).$$

Applying Kramer's method, we obtain a solution in the form of the following autonomous system of differential equations:

$$\frac{d\bar{V}}{d\bar{\eta}}=-\frac{Bb\left(1+\bar{p}'\right)}{\left(1+\alpha_1\bar{p}'\right)\left(\bar{V}-1\right)^2-\left(1+\bar{p}'\right)}, \qquad (24)$$

$$\frac{d\bar{p}'}{d\bar{\eta}}=\frac{Bb\left(\bar{V}-1\right)\left(1+\alpha_1\bar{p}'\right)\left(1+\bar{p}'\right)}{\left(1+\alpha_1\bar{p}'\right)\left(\bar{V}-1\right)^2-\left(1+\bar{p}'\right)}. \qquad (25)$$

The system of equations (24), (25) can be solved directly, but there is a special approach. Due to the autonomy of this system, it is possible to use the phase plane and consider the pressure disturbance as a function of speed:

$$\frac{d\bar{p}'}{d\bar{V}}=-\left(\bar{V}-1\right)\left(1+\alpha_1\bar{p}'\right). \qquad (26)$$

The general solution of equation (26) is as follows:

$$\bar{p}'\left(\bar{V}\right)=C_1\exp\left(-\alpha_1\bar{V}\left(\bar{V}-2\right)\right)-1/\alpha_1. \qquad (27)$$

Obviously, for a compression wave, the pressure perturbation must be positive, so in the solution (27) $C_1>0$. The maximum pressure perturbation is reached for the value $\bar{V}=1$ and corresponds to

$$\bar{p}'\left(\bar{V}=1\right)=C_1\exp\left(\alpha_1\right)-1/\alpha_1. \qquad (28)$$

From relation (28), it is quite easy to understand what the unknown constant of integration is. Indeed, we recall that the scale of pressure disturbances was chosen to be its maximum. This means that

$$\bar{p}'\left(\bar{V}=1\right)=C_1\exp\left(\alpha_1\right)-1/\alpha_1=1.$$

Here $C_1=\left(1+1/\alpha_1\right)\exp(-\alpha_1)$.

Therefore, the final form of the pressure disturbance function is as follows:

$$\bar{p}'\left(\bar{V}\right)=\left(1+1/\alpha_1\right)\exp\left(-\alpha_1\left(\bar{V}-1\right)^2\right)-1/1/\alpha_1. \qquad (29)$$

Expression (29) clearly indicates that everywhere outside the shock pulse maximum, where $\bar{V}\neq 1$, pressure disturbances are smaller than the maximum value. In addition, from Fig. 1, a it can also be seen that the graph of the curve of pressure dependence on velocity is qualitatively similar to Fig. 2 [1], as well as Fig. 2-4 [4]. However, pressure disturbances cannot take any value. Therefore, the range of possible values of the shock wave propagation speed is limited. These restrictions are determined by the maximum possible negative value of gauge pressure disturbances: they should not, in absolute terms, exceed the value of the working pressure in the pipeline, because the absolute pressure cannot take negative values. Even more – for a liquid, it is the pressure threshold at which the phenomenon of cavitation occurs. Therefore, one should look for a limit on the speed of shock wave propagation from the following inequality:

$$\left[\bar{p}'\right]\bar{p}'\left(\bar{V}\right)=\left[\bar{p}'\right]\left(2\exp\left(-\alpha_1\left(\bar{V}-1\right)^2/2\right)-1\right)>-p_w+p_{ss}.$$

In the last ratio p_w, p_{ss}, respectively, the working pressure in the pipeline and the pressure of saturated steam.

From the system of equations (24), (25) it is possible to find $\bar{V}\left(\bar{\eta}\right)$ and $\bar{p}'\left(\bar{\eta}\right)$. Let's start with equation (24) and substitute solution (29) into it. As a result, the following equation is obtained relative to $\bar{V}\left(\bar{\eta}\right)$:

$$\frac{d\bar{V}}{d\bar{\eta}} = -Bb\left(1-\frac{1}{\alpha_1}+\left(1+\frac{1}{\alpha_1}\right)\exp\left(-\alpha_1\left(\bar{V}-1\right)^2\right)\right)/$$
$$/\exp\left(-\alpha_1\left(\bar{V}-1\right)^2\right)\left(\left(1+\alpha_1\right)\left(\bar{V}-1\right)^2-\left(1+\frac{1}{\alpha_1}\right)\right)- \quad (30)$$
$$-1\left(1-\frac{1}{\alpha_1}\right).$$

Ratio (30) clearly indicates the complex nature of the solution for the velocity function in the case of direct integration of equation (30). On the basis of equation (30), an analytical solution can be obtained if we consider equation (30) in its inverse form:

$$\frac{d\bar{\eta}}{d\bar{V}} = -\exp\left(-\alpha_1\left(\bar{V}-1\right)^2\right)\left(\left(1+\alpha_1\right)\left(\bar{V}-1\right)^2-1-\frac{1}{\alpha_1}\right)-$$
$$-1\left(1-\frac{1}{\alpha_1}\right)/\,Bb\left(1-\frac{1}{\alpha_1}+\left(1+\frac{1}{\alpha_1}\right)\exp\left(-\alpha_1\left(\bar{V}-1\right)^2\right)\right).$$

After algebraic transformations, it is possible to obtain a solution in the form

$$\bar{\eta}\left(\bar{V}\right) = -\frac{\alpha_1}{Bb}\frac{\left(\bar{V}-1\right)^3}{3}+\frac{\bar{V}}{Bb}+C_1+$$
$$+\int\frac{\alpha_1\left(\alpha_1-1\right)\left(\bar{V}-1\right)^2}{Bb\left(\left(1+\alpha_1\right)\exp\left(-\alpha_1\left(\bar{V}-1\right)^2\right)+\alpha_1-1\right)}dv. \quad (31)$$

To evaluate the significance of the term in (31) containing the integral, we should return to the definition of the parameter Bb. Let's make an assessment:

$$\frac{\alpha_1-1}{Bb}=\frac{\alpha_1-1}{3\alpha_2}\frac{\alpha_1}{1}\frac{c_1}{w_{1a}}\frac{a}{L}=\frac{\alpha_1}{3}\frac{c_1}{w_{1a}}\frac{a}{L}. \quad (32)$$

The right-hand side in (32) is the product of three ratios, two of which have the order of unity, but the third, the ratio of the bubble radius to the shock wave length, is obviously very small. Indeed, taking into account the speed of wave propagation (about 1000 m/s) and the frequency of the hydraulic shock wave (tens to few hundreds of Hertz), we have a wavelength of the order of meters or more, which is obviously many times greater than the radius of the bubble (millimeters). So, with fairly high accuracy, it can be assumed

$$\bar{\eta}\left(\bar{V}\right) = -\frac{\alpha_1}{Bb}\frac{\left(\bar{V}-1\right)^3}{3}+\frac{\bar{V}}{Bb}+C_1. \quad (33)$$

The integration constant C_1 in the solution (33) is easily determined from the boundary condition:

$$\bar{\eta}\left(\bar{V}=1\right)=0. \quad (34)$$

It follows from relations (33) and (34) that

$$\bar{\eta}\left(\bar{V}\right)=\frac{1}{Bb}\left(-\frac{\alpha_1}{3}\left(\bar{V}-1\right)^3+\bar{V}-1\right). \quad (35)$$

From equation (35) it is already easy to find the inverse function $\bar{V}=\bar{V}(\bar{\eta})$. The only one of the three solutions (35) is real:

$$\bar{V}(\bar{\eta})=\varphi(Bb,\bar{\eta})/\alpha_1-1/\varphi(Bb,\bar{\eta}), \quad (36)$$

with

$$\varphi(Bb,\bar{\eta})=\frac{1}{2}\left(\left(12Bb\bar{\eta}+4\sqrt{\frac{9Bb^2\alpha_1\bar{\eta}^2-4}{\alpha_1}}\right)\alpha_1^2\right)^{1/3}.$$

But it does not exist for all values of $\bar{\eta}$. The solution (36) is obtained by Cartano's formulas. In the case when the expression inside the square root is negative, one can use the trigonometric approach to the solution [20, see (1.8.8)]. We will briefly indicate how to do it. Equation (35) can be rewritten as

$$u^3+pu+q=0,\ u=\bar{V}\text{-}1,\ p=-3/\alpha_1,\ q=3Bb\bar{\eta}/\alpha_1. \quad (37)$$

Equation (37) has three valid solutions [20, see (1.8.8)]:

$$u_1=2\sqrt{-p/3}\cos(\alpha/3),$$
$$u_{2,3}=-2\sqrt{-p/3}\cos(\alpha/3\pm\pi/3). \quad (38)$$

In formulars (38)

$$\cos(\alpha)=q/(2\sqrt{-(p/3)^3}.$$

The graphical dependence $\bar{V}=\bar{V}(\bar{\eta})$, according to (38), is shown in Fig. 1, b. Comparing it with Fig. 11 of work [6] indicates a good correspondence.

Obviously, the direct solution of the system (26), (27) will lead to too cumbersome expressions for the pressure disturbance function $\bar{p}'(\bar{\eta})$. Therefore, it is necessary to just numerically solve the system of equations (24), (25), which is done below as a particular case – without taking into account the friction of the liquid against the walls.

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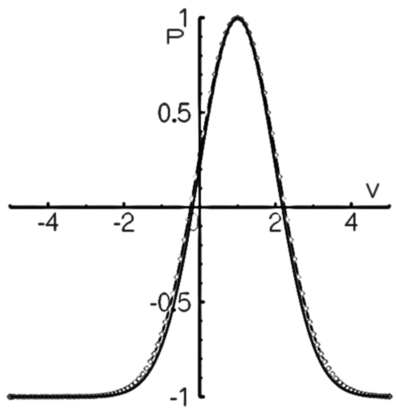

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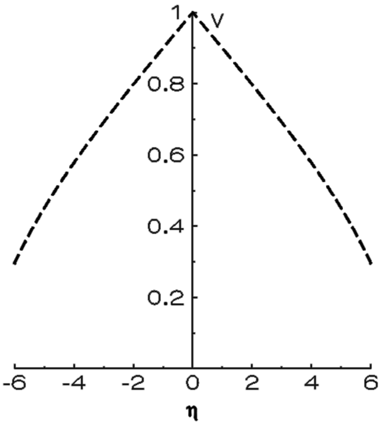


b

Fig. 1 a – Dependence of pressure disturbances on the speed of propagation of the shock pulse: the solid line corresponds to a homogeneous liquid, and the markers correspond to a bubble according to the value of the parameter $\alpha_1 = 0.9$;
b – propagation velocity in the shock pulse

4.2. Taking into account a friction against the pipe wall

Let us now also take into account the friction of the liquid against the pipe wall – the classic Weissbach-Darcy model [18, 19]. As mentioned above, with a small concentration of bubbles, this friction can be roughly approximated by the same model as in a homogeneous liquid. The matrix of the system (21), (22) looks like this:

$$\begin{pmatrix} (1+\alpha_1\bar{p}')(\bar{V}-1) & 1 & \left|\bar{\lambda}(1+\bar{p}')\bar{V}|\bar{V}|\right. \\ (1+\bar{p}') & (\bar{V}-1) & \left|\mathrm{Bb}(1+\bar{p}')\right. \end{pmatrix} \quad (39)$$

The determinant of the matrix has not changed, but Δ_1 they Δ_2 already look different:

$$\Delta_1 = (1+\bar{p}')\left(\bar{\lambda}(\bar{V}-1)\bar{V}|\bar{V}| - \mathrm{Bb}\right),$$
$$\Delta_2 = (1+\bar{p}')\left(\mathrm{Bb}(1+\alpha_1\bar{p}')(\bar{V}-1) - \bar{\lambda}\bar{V}|\bar{V}|(1+\bar{p}')\right).$$

According to Kramer's method, we obtain the following system of differential equations:

$$\frac{d\bar{V}}{d\bar{\eta}} = \frac{\left(\bar{\lambda}(\bar{V}-1)\bar{V}|\bar{V}| - \mathrm{Bb}\right)(1+\bar{p}')}{(1+\alpha_1\bar{p}')(\bar{V}-1)^2 - (1+\bar{p}')}, \quad (40)$$

$$\frac{d\bar{p}'}{d\bar{\eta}} = \frac{(1+\bar{p}')\left(\mathrm{Bb}(1+\alpha_1\bar{p}')(\bar{V}-1) - \bar{\lambda}\bar{V}|\bar{V}|(1+\bar{p}')\right)}{(1+\alpha_1\bar{p}')(\bar{V}-1)^2 - (1+\bar{p}')}. \quad (41)$$

We use the experience of solving the previous problem and we find it first $\bar{p}' = \bar{p}'(\bar{V})$, only later $\bar{\eta} = \bar{\eta}(\bar{V})$. Dividing the left and right parts of equation (41) by correspondent ones of equation (40), we obtain:

$$\frac{d\bar{p}'}{d\bar{V}} = \frac{\mathrm{Bb}(1+\alpha_1\bar{p}')(\bar{V}-1) + \bar{\lambda}\bar{V}^2(1+\bar{p}')}{-\bar{\lambda}(\bar{V}-1)\bar{V}^2 - \mathrm{Bb}}, \quad \bar{V} \le 0; \quad (42)$$

$$\frac{d\bar{p}'}{d\bar{V}} = \frac{\mathrm{Bb}(1+\alpha_1\bar{p}')(\bar{V}-1) - \bar{\lambda}\bar{V}^2(1+\bar{p}')}{\bar{\lambda}(\bar{V}-1)\bar{V}^2 - \mathrm{Bb}}, \quad \bar{V} \ge 0. \quad (43)$$

The corresponding general solutions of equations (42) and (43) are as follows:

$$\bar{p}'(\bar{V} \le 0) = \left(-\int \varphi_1 \frac{\bar{\lambda}\bar{V}^2 + \mathrm{Bb}\bar{V} - Bb}{-\bar{\lambda}\bar{V}^3 + \bar{\lambda}\bar{V}^2 + \mathrm{Bb}} d\bar{V} + C_1\right)\varphi_1^{-1}, \quad (44)$$

$$\bar{p}'(\bar{V} \ge 0) = \left(\int \varphi_1 \frac{\bar{\lambda}\bar{V}^2 - \mathrm{Bb}\bar{V} + Bb}{-\bar{\lambda}\bar{V}^3 + \bar{\lambda}\bar{V}^2 + \mathrm{Bb}} d\bar{V} + C_1\right)\varphi_1^{-1}, \quad (45)$$

with $\varphi_1(\bar{V} \le 0) = \int \frac{\alpha_1\bar{\lambda}(\bar{V}-1)}{\bar{\lambda}\bar{V}^3 - \bar{\lambda}\bar{V}^2 + \mathrm{Bb}} d\bar{V},$

$$\varphi_1(\bar{V} \ge 0) = \int \frac{\alpha_1\bar{\lambda}(\bar{V}-1)}{-\bar{\lambda}\bar{V}^3 + \bar{\lambda}\bar{V}^2 + \mathrm{Bb}} d\bar{V}.$$

In addition to solutions (44), (45) in the form of integrals, one can also find approximate solutions in the form of series for $|\bar{V}-1| \le (\approx 0.5)$. Given the smallness of the values Bb and the boundary condition $\bar{p}'(\bar{V}=1) = 0$, the following approximation is obtained for $\bar{V} \ge 0$

$$\bar{p}'(\bar{V}) = 1 + \frac{2\bar{\lambda}}{1+\mathrm{Bb}}(\bar{V}-1) - \frac{\mathrm{Bb}(\alpha_1 - 8\bar{\lambda} + 1) - 4\bar{\lambda}^2 - 4\bar{\lambda}}{2(1+\mathrm{Bb})^3}(\bar{V}-1)^2 -$$

$$-\frac{\bar\lambda}{6}\frac{\left(\mathrm{Bb}\left(12\bar\lambda^2+7\alpha_1-60\bar\lambda-13\right)-12\bar\lambda^2-28\bar\lambda-4\right)}{(1+\mathrm{Bb})^5}\left(\bar V-1\right)^3-$$

$$-\frac{\bar\lambda}{24}\frac{\left(\mathrm{Bb}\left(144\bar\lambda^3+40\alpha_1\bar\lambda-248\bar\lambda^2+30\alpha_1-468\bar\lambda+18\right)\right.}{(1+\mathrm{Bb})^7}-$$

$$\left.-48\bar\lambda^3-184\bar\lambda^2-112\bar\lambda\right)\frac{\bar\lambda}{24}\frac{\left(\bar V-1\right)^4}{(1+\mathrm{Bb})^7}\left(\bar V-1\right)^4-$$

$$-\frac{\bar\lambda}{120}\frac{1}{(1+\mathrm{Bb})^9}\left(\mathrm{Bb}\left(1440\bar\lambda^5+248\alpha_1\bar\lambda^2+512\bar\lambda^3-240\bar\lambda^4+\right.\right.$$

$$+468\alpha_1\bar\lambda-5828\bar\lambda^2-1304\bar\lambda^3+52\alpha_1-1558\bar\lambda-$$

$$\left.\left.-1600\bar\lambda^2+36\right)-242\bar\lambda\right)\left(\bar V-1\right)^5+O\left(\left(\bar V-1\right)^5\right). \qquad (46)$$

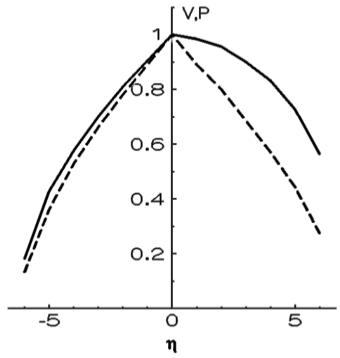
For the values $\bar V \le 0$ and the boundary condition $\bar p'\left(\bar V=0\right)=0$ the following approximation is obtained:

$$\bar p'\left(\bar V\right)=\mathrm{Bb}\bar V-\frac{1}{2}\mathrm{Bb}\bar V^2+\frac{1}{3}(\mathrm{Bb}+1)\bar\lambda\bar V^3-\frac{1}{12}\mathrm{Bb}\bar\lambda\left(\alpha_1+9\right)\bar V^4+$$

$$+\frac{1}{5}\left(\frac{\mathrm{Bb}\bar\lambda}{3}\alpha_1+\mathrm{Bb}\bar\lambda^2-\bar\lambda^2+\frac{3}{2}\mathrm{Bb}\bar\lambda\right)\bar V^5+O\left(\bar V^6\right). \qquad (47)$$

5. Comparative analysis of solutions for homogeneous and bubbly liquids

As mentioned in the introduction, the presence of gas bubbles is responsible for the compressibility of bubbly liquid [1]. That is probably why in the models of water hammer in bubbly liquid special attention is not paid to the elastic deformations of the pipeline. Although this topic can be discussed (see the Discussion section below). Let us now focus on the results obtained by numerical integration of the system of equations (40), (41). Fig. 2 presents the corresponding values of the functions $\bar p'(\bar\eta)$ and $\bar V(\bar\eta)$. The peculiarity of these curves is that their left parts correspond to the process of stretching bubbles, and the right parts correspond to compression. Significant displacement of the maximum $\bar p'(\bar\eta)$ (see Fig. 2, d from the center indicates the contradiction of the assumption of a decreasing function $\bar p'(\bar\eta)$ for all positive values of the argument. By fitting, it was found that for $\bar\lambda = 0.025$ a fairly physical picture is obtained, when the maximum of the pressure disturbance field almost does not move, and the velocity distribution $\bar V(\bar\eta)$ very well resembles its counterpart in the case of a homogeneous liquid (see Fig. 1 [17]). The only difference is that it became clear after comparing with the solution (23) (see [17]) that the integration constants in the cited solution can be chosen so that the graph $\bar V(\bar\eta)$ has a qualitative similarity with Fig. 3.



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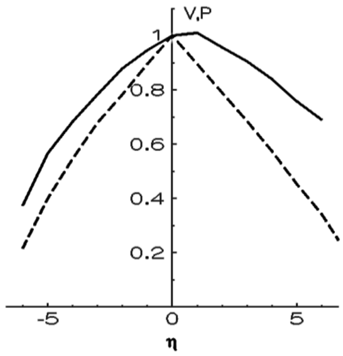

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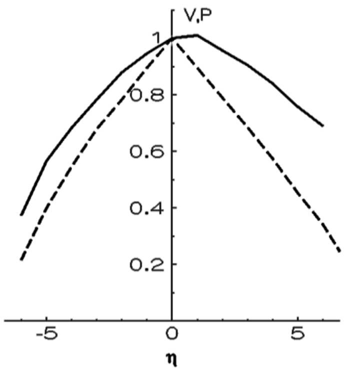

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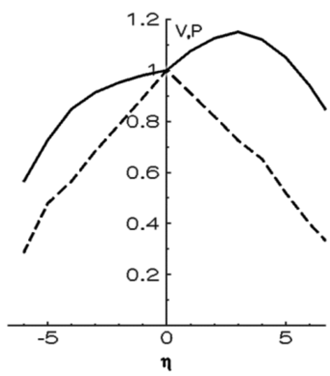

d

Fig. 2. Dependencies of the velocity field and pressure disturbance. $|\mathrm{Bb}|=0.1$ (everywhere):
a – $\bar\lambda=0.01$; b – $\bar\lambda=0.01$;
c – $\bar\lambda=0.05$; d -- $\bar\lambda=0.1$

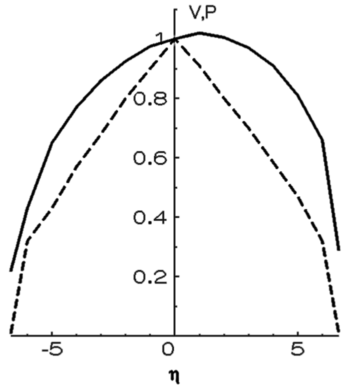

Fig. 3. Distribution of pressure disturbances and the velocity field, which are most consistent with those obtained in previous work [17]. The parameter data are as follows: $|Bb| = 0.1$, $\bar{\lambda} = 0.025$

Discussion

The phenomenon of water hammer is quite complex. It refers to unsteady fluid flows. The peculiarity of the passage of a shock pulse in a bubbly liquid is that the compression of the medium occurs mainly due to the deformation of gas bubbles. The relations given in the work characterizing the bubbly liquid as a one-speed continuum are well-known [15]. Mathematical complexity caused the consideration of small perturbations that corresponded to linear models. However, pressure changes occurring in flexible pipelines on airplanes and helicopters reach 75% of the operating value. In such circumstances, the assumption of smallness has no place. So, nonlinear models are considered in the work. The presence of gas impurities in the liquid is quite likely, since cavitation phenomena occur in the hydraulic system, during which a small amount of air may enter through the seal. Therefore, it is advisable to take into account the bubbliness of the liquid, which is confirmed by significant differences of the investigated in the work processes, respectively, in homogeneous and bubble liquids. From the theoretical (physical) point of view, in the bubbly liquid model, the radial deformation rate of the bubbles is considered a constant value. At first view, one may think that this is not the case, but the analysis of graphic information in Fig. 2 and Fig. 3 shows that the divergence of the velocity field, which is physically responsible for fluid compressibility, is approximately constant (the velocity graphs are quite close to straight lines). Hence, the compression-expansion rates are also approximately constant, which validates the model. Another assumption used in this work is the slight difference in Weisbach-Darcy friction for homogeneous and bubbly liquids. And indeed, physically consistent results are obtained only for rather small (0.025) dimensionless value of the friction parameter.

Conclusions

The paper formulated and solved the problem of unsteady flow (shock wave propagation) of droplet liquid containing a small amount of bubbles (bubble liquid). The application of the original method for solving the nonlinear system of equations of the model made it possible to find an analytical solution - the dependence of the of pressure field disturbances on the speed of the shock pulse. Quantitative analysis indicates that pressure changes of 100-200 atm. do not exceed 1% of the theoretical maximum permissible (2GPa).

Another possible unsteady flow can be one where the velocity distribution is significantly variable in space and can reach a zero value. In fact, this means "blurring" of the shock pulse. The obtained data are consistent with previous results [17]. The shape of the speed distribution (see Fig. 3) resembles the corresponding shape presented in Fig. 1 in [17] for a homogeneous liquid. The difference is that for a bubbly liquid, the area of the shock pulse is 10-12 times longer than that for a homogeneous liquid.

As further research, it is possible to improve the model, as well as use the results obtained in this work for more accurate modeling of the formation and propagation of a shock pulse in a multiphase medium, which is a bubbly liquid.

Contribution of authors: conceptualization – **Pavlo Lukianov**, **Katerina Pavlova**; formulation of task – **Pavlo Lukianov**; analysis – **Pavlo Lukianov**, **Katerina Pavlova**; software – **Katerina Pavlova**; development of mathematical model – **Pavlo Lukianov, Katerina Pavlova;** analysis of results – **Pavlo Lukianov**, **Katerina Pavlova**.

Conflict of interest

The authors declare that they have no conflict of interest in relation to this research, whether financial, personal, authorship or otherwise, that could affect the research and its results presented in this paper.

Financing

The study was conducted without financial support.

Data availability

The Manuscript has no associated data.

Use of Artificial Intelligence

The authors confirm that they did not use artificial intelligence methods while creating the presented work.

All the authors have read and agreed to the published version of this manuscript.

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НЕСТАЦІОНАРНА ТЕЧІЯ БУЛЬБАШКОВОЇ РІДИНИ В ГІДРАВЛІЧНИХ СИСТЕМАХ ЛІТАКІВ І ВЕРТОЛЬОТІВ

П. В. Лук'янов, К. С. Павлова

Предметом даної роботи є явище гідравлічного удару в рідини, яка містить незначну за об'ємом частину бульбашок газу. Історично це явище почалося вивчатися як динаміка бульбашок газу (рівняння Релея-Плесета). І сьогодні, завдяки прогресу у комп'ютерній техніки, це явище вивчається на рівні деформації бульбашки при проходженні гідравлічного удару. Інший підхід представляє собою розгляд динаміки багатофазного (двофазного) середовища у вигляді бульбашкової рідини. Після низки припущень, основним із яких є відносно малий вміст газу в рідині, модель складається з двох диференціальних рівнянь відносно швидкості поширення ударної хвилі та збурень тиску, які виникають внаслідок цього. У зазначеній системі рівнянь є відмінність від відповідних класичних рівнянь гідроудару: вони враховують конвекцію поля швидкості. Крім того, враховується тертя рідину о стінку за моделлю Вейсбаха-Дарсі. Зважаючи на невеликий вміст бульбашок газу, тертя Вейсбаха-Дарсі апроксимується так само як і в однорідній рідині, тобто в певному сенсі більшим ніж реальне. Може тому, лише за невеликих значень безрозмірного параметру, що відповідає за тертя рідини о стінку, отримуються більш-менш фізичні результати. Йдеться про несуперечність припущень и результатів, які отримуються на їх підставі. Так у передній області ударного імпульсу, де відбувається підвищення тиску, радіальна швидкість у бульбашок є від'ємною, але за відносно великих значень параметра тертя максимум збурень тиску переміщується з центру ударного імпульсу. І це протирічить припущенню про стиснення: після проходження максимуму тиску відбувається розширення бульбашок газу – за рахунок зменшення тиску. Отримані в роботі графічні залежності порівняні із результатами, що стосуються однорідної рідини. Вони узгоджуються, але ударний імпульс в бульбашковій рідині не такий концентрований у просторі як у однорідній. Його протяжність у 10-12 разів перевищує відповідне значення у однорідній рідині. **Методи досліджень** є суто теоретичними. Використовується відома модель бульбашкової рідини як одношвидкістного континуму. Диференціальні рівняння розв'язуються аналітично, наближено (розвиненням у ряди) та чисельно. Крім того, застосовується оригінальний підхід отримання аналітичного розв'язку автономної системи – знаходження функції збурень тиску від швидкості поширення ударного імпульсу (та навпаки). **Висновки.** Запропонована проста одновимірна гідравлічна модель поширення ударної хвилі (імпульсу) у бульбашковій рідині. На відміну від класичних уявлень (розв'язків) про гідравлічний удар, який складається із двох хвиль протилежних напрямків поширення, ударний імпульс являє собою область збурень тиску, в якій швидкість руху частинок рідини також є змінною – від максимального значення до майже нульового.

Ключові слова: літак; вертоліт; елемент конструкції; гідравлічний удар; двофазна течія; напруження; деформація поверхні; втома.

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