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Pavlo LUKIANOV, Kateryna PAVLOVA

National Aviation University, Kyiv, Ukraine

UNSTEADY FLOW OF DROPLET LIQUID IN HYDRAULIC SYSTEMS OF AIRCRAFTS AND HELICOPTERS: MODELS AND ANALYTICAL SOLUTIONS

The subject *of this study is the unsteady flow of liquid in pipelines, which are part of the design of airplanes and helicopters. This name means, first of all, the phenomenon of a sharp increase in pressure in the pipeline, which is known as a hydraulic shock. Although we have already learned to deal with this phenomenon in some parts of the systems, in many structural elements (flexible pipelines), inside which the working pressure reaches several hundred atmospheres, this phenomenon is still quite dangerous. As you know, the best way to deal with an unwanted phenomenon is through theoretical study. To date, there has been a huge amount of work in the direction of hydraulic shock research. This article does not fully cover these studies. It is limited to references to reviews and relevant works. Because the phenomenon of hydraulic shock has a significantly nonlinear character, analytical solutions of systems of equations corresponding to the simplest models were unknown until recently. This work presents, as an overview, already known analytical solutions describing the process of shock wave propagation. Most importantly, new achievements are given, both for the inviscid approximation and for considering internal viscous friction. It is shown that the internal friction within the considered model is negligible almost everywhere, except for the thin shock layer. The asymptotic is proportional to the tangent function and inversely proportional to the square root of the product of the Reynolds number and the dimensionless parameter characterizing the convection effect. Convection of the velocity field significantly affects the distribution of characteristics in hydraulic shock. If the self-similar solutions that were obtained earlier have a power-law character for the velocity distribution in the shock wave, then the simultaneous consideration in the model of convection and friction on the pipeline walls (according to the Weisbach-Darcy model) made it possible to obtain a distribution in the form of an exponential function that decays with increasing distance from the shock wave front. In addition, the work includes an original approach to solving a nonlinear system of differential equations that describes the propagation of a shock wave without considering the friction on the walls. Analytical solutions were obtained in the form of a function of pressure versus the velocity of shock wave propagation.* ***Research methods.*** *This work uses purely theoretical approaches based on the use of well-known fluid flow models, methods of analytical solution of differential equations and their systems, asymptotic methods, derivation of self-model equations, and finding their solutions.* ***Conclusions.*** *Analytical solutions of systems of differential equations were obtained, which describe models of hydraulic shock without considering viscous effects. A comparison of the obtained results with the results of other studies is given.*

Keywords: *plane; helicopter; structural elements; hydraulic shock (shock wave); stress; friction; surface deformation; fatigue.*

Introduction

The phenomenon of non-stationarity of the fluid flow, which leads to the formation of a shock wave, occurs at flight speeds close to the speed of sound [1] and greater than the speed of sound [2]. When designing hydraulic and other systems of airplanes and helicopters, one should take into account the fact that during flight in structural elements, which are both relatively rigid and flexible pipelines, an undesirable phenomenon of water hammer may occur. From a hydrodynamic point of view, this is essentially an unsteady flow of a droplet liquid in a closed volume.

Already at the end of the 60s - beginning of the 70s of the last century, very high working pressures began to be used in aviation: 200-300 atm. Thus, at a working pressure of 200 atm., an increase up to 350 atm was observed during water hammer [3]. Therefore, in [4], a nonlinear model of shock wave propagation was considered. In this model, in contrast to [3], the nonlinear mechanism of friction of the liquid against the pipe wall is already taken into account. But further theoretical studies of the phenomenon of hydraulic shock in a two-phase fluid revealed some discrepancies in modeling approaches. This determined (became the generator of) this work.

In the 1860s, Riemann managed to show that there is a special solution to the system of gas-thermodynamic equations. This solution does not belong to the class of general or partial solutions. Instead, it represents a break in the pressure, density, and velocity functions of the medium, i.e., a sharp jump in parameters that has no width.

Later, in 1885-1887, there appeared works [5, 6] devoted to hydraulic shock. But the works of Joukowski [7] and Allievi [8] attracted considerable attention of the scientific community.

In his theory, Allievi took into consideration the speed of propagation of the shock wave a [8]:

$$\frac{1}{a^2}=\frac{\omega}{g}\left(\frac{1}{\varepsilon}+\frac{1}{E}\frac{D}{e}\right). \tag{1}$$

In formula (1) $E, \varepsilon, D, e, \omega$, the modulus of elasticity of the pipe, the modulus of elasticity of the liquid, the diameter of the pipeline, the thickness of the pipeline, and the density of the liquid mean, respectively. For relatively thin steel pipes, the speed of wave is approximately 600-700 m/s. And for relatively thick pipes, its value reaches 1200-1300 m/s. Since the speed of sound c in water varies within $1403<c<1555$ m/s, it means that, in an elastic shell, a shock wave propagates at a speed lower than the speed of sound in a liquid.

According to Alliveti, the pressure in the form of a column and the shock wave propagation speed are described by the following relations [8]:

$$y=y_0+F+f, \tag{2}$$

$$V=V_0+F-\frac{g}{a}(F-f). \tag{3}$$

At the same time, the arguments of the functions are such that

$$F=F\left(t-\frac{x}{a}\right),\ f=f\left(t+\frac{x}{a}\right). \tag{4}$$

A simple conclusion follows from expressions (4):

both functions have a constant value if their arguments are constants. And this is possible under the conditions of propagation of direct and opposite shock waves with a speed of a. The given data already have, as shown below, their meaning, because they indicate the form of self-similar variable selection.

In his theoretical studies, Joukowski [7] refers to Riemann's work [9], where the general structure of the solution according to (4) is shown. It should be noted that in the model considered by Joukowski, the convective term in the equation of conservation of momentum (momentum) was not neglected. The fact is that the problems considered by Alliveti and others at the end of the 19th century were devoted to pipelines – long pipes. Therefore, the length scale was such that it allowed Alliveti to neglect the convective term. On the other hand, shock wave in the structural elements of aviation equipment does not necessarily occur in very long pipelines. For more than half a century, flexible connections in the form of rather short tubes have been used in aviation. On the other hand, as experiments show (let's take at least the same work by Joukowski [7]), the shock wave practically decays at a distance of the order of ten diameters of the pipe. Therefore, it is impossible to neglect the longitudinal gradient of the flow. At the end of a brief review of Joukowski's work, it should be noted that it presents the forms of the velocity distribution that are similar to the function of the square root of the coordinate. This is the form of one of the solutions obtained further in this paper.

Despite the significant number of works in the theory of hydraulic shock (water hammer) [10], let's pay attention to those that continued the development of the ideas of Alliveti and Joukowski. Thus, the analytical solution of the system that takes into account Weisbach - Darcy model for wall friction was obtained relatively recently [11]

$$\frac{\partial V}{\partial x}+\frac{1}{a^2\rho_0}\frac{\partial p}{\partial t}=0, \tag{5}$$

$$\frac{\partial V}{\partial t}+\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\frac{\lambda}{4R}V|V|-g\sin(\varphi)=0, \tag{6}$$

where x, t – the direction of wave propagation and time;, V, p – speed and pressure in the liquid, ρ_0, a, λ – respectively, the density of the liquid, the speed of sound propagation in it, and the coefficient of friction; R, g – hydraulic radius, free fall acceleration, and φ – the angle of inclination of the pipeline to the horizon.

In the model described by equations (5) and (6), the effect of friction on the propagation of the shock wave has already been taken into account. It uses the hydraulic approximation – the Weisbach-Darcy viscous friction model [12, 13].

The solution of system (5) and (6) is [11]:

$$V(x,t)=C^{-1}\tanh\left(\operatorname{artanh}(V_0C)-xD\right),\quad p(x,t)=\frac{P_0}{V_0}V(x,t),$$

where $C=\frac{1}{2}\sqrt{\frac{\lambda}{Rg\sin\varphi}}$,

$$D=\frac{p_0\rho_0V_0}{2\left(c\rho^3V_0^2-p_0^2\right)}\sqrt{\frac{g\sin\varphi\lambda}{R}}.$$

Three types of self-similar solutions of the system (5) and (6) in the case of a horizontal pipeline were also obtained in [4].

Self-similar (general, with two constant of integration) solution of the first kind:

$$V(\eta)=\tanh\left[\frac{\sqrt{C_1\left(a^2-1\right)a/2}\,(\eta+C_2)}{a^2-1}\right]\cdot\sqrt{2C_1\left(a^2-1\right)/a}\,/\,\theta.$$

Self-similar solution of the second kind:

$$V(\eta)=\frac{\left(1\text{-}a^2\right)V_0}{1\text{-}a^2+aV_0\eta}. \quad (7)$$

Self-similar (general, with one constant of integration) solution of the third kind:

$$V(\eta)=\frac{(2a-1)\left(1+a^2\right)}{\eta^{1-\alpha}+C_1\left(1+a^2\right)(2a-1)\eta^{\alpha}}. \quad (8)$$

In addition to those three self-similar solutions, an analytical solution of the problem was obtained in [4] as well. This (general, with three constants of integration) solution in dimensionless coordinates (indicated by a dash from above) has the form:

$$\bar{V}\left(\bar{x},\bar{t}\right)=\frac{2\left(C_3^2\text{-}C_2^2\right)\tanh\left[C_1+C_2\bar{x}+C_3\bar{t}\right]}{C_3\theta}.$$

An interesting fact is that partial solutions (7) and (8) make it possible to obtain discontinuous functions, which is in agreement with [14, 15] (see [4], Fig. 1). Other solutions have the form of a blurred jump with the speed decaying away from this jump (see [4], Fig. 2, a). As shown below, this result is close to the solution obtained in this paper taking into account the convective term. Therefore, taking into account the friction of the liquid against the wall leads to a smearing of the velocity profile. In addition, the profile of the shock wave can have not only a smooth character (hyperbolic tangent), but also the appearance of a sharp, almost instantaneous increase, which changes to a decrease. All this testifies to the diversity and physical complexity of the phenomenon of hydraulic shock (unsteady flow) in a droplet liquid.

Problem formulation

1. Study the effects of convection and internal friction in the unsteady flow of a droplet liquid.
2. Obtain analytical solutions of the problem and make a comparative analysis of them with already known solutions given in the introduction, in which only the friction of the liquid against the wall is taken into account in the hydraulic approximation (the Weisbach-Darcy friction model).
3. Obtain the function that allows to find the pressure value based on the values of the wave propagation speed.

2. Shock wave model that takes into account the velocity field convection

2.1. Derivation of the equation in dimensional and dimensionless forms

If, when considering an unsteady flow, one does not take into account viscosity, but only convection, then the corresponding system of equations of conservation of momentum and mass has the following form:

$$-\frac{\partial p}{\partial x}=\rho_0\left(\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}\right); \quad (9)$$

$$-\frac{\partial p}{\partial t}=a^2\rho_0\frac{\partial V}{\partial x}. \quad (10)$$

To exclude pressure, p we take the partial derivatives of each of the equations. The first – by time, the second – by the x coordinate. After that, the system of equations (9) and (10) will turn into the following:

$$-\frac{\partial^2 p}{\partial x\partial t}=\rho_0\left(\frac{\partial^2 V}{\partial t^2}+\frac{\partial V}{\partial t}\frac{\partial V}{\partial x}+V\frac{\partial^2 V}{\partial x\partial t}\right); \quad (11)$$

$$-\frac{\partial^2 p}{\partial t\partial x}=a^2\rho_0\frac{\partial^2 V}{\partial x^2}. \quad (12)$$

From equations (11) and (12), excluding pressure, we have:

$$a^2\frac{\partial^2 V}{\partial x^2}=\frac{\partial^2 V}{\partial t^2}+\frac{\partial V}{\partial t}\frac{\partial V}{\partial x}+V\frac{\partial^2 V}{\partial x\partial t}. \quad (13)$$

For an infinite domain, equation (13) is supplemented by two initial conditions [4]:

$$V(x,0)=\varphi(x); \qquad \frac{\partial V}{\partial t}(x,0)=\psi(x). \quad (14)$$

So, the task was reduced to finding a solution to equation (13) that satisfies the initial conditions (14). For a qualitative analysis of the solutions, let us switch to the dimensionless form of equation (13). For this purpose, we formally introduce the scales of length and time, the scale of speed, etc. [4]:

$$[x]=L, \quad [t]=T, \quad [V]=L/T. \qquad (15)$$

We consider the balance of forces, assuming that the terms of the linear wave equation have the same (main) order of magnitude:

$$\left[a^2\frac{\partial^2 V}{\partial x^2}\right]=\left[\frac{\partial^2 V}{\partial t^2}\right]. \qquad (16)$$

Based on relations (15), we obtain the following expressions:

$$\left[a^2\frac{\partial^2 V}{\partial x^2}\right]=\frac{a^2 L}{TL^2}; \qquad \left[\frac{\partial^2 V}{\partial t^2}\right]=\frac{L}{T^3}. \qquad (17)$$

Substitution (17) in (16) makes it possible to obtain

$$\frac{a^2 L}{TL^2}=\frac{L}{T^3} \qquad \Rightarrow \qquad a=\frac{L}{T}.$$

Taking into account the obtained scale estimates, equation (13) is transformed to the following dimensionless form:

$$\frac{\partial^2\bar{V}}{\partial\bar{x}^2}=\frac{\partial^2\bar{V}}{\partial\bar{t}^2}+\theta\left(\frac{\partial\bar{V}}{\partial\bar{t}}\frac{\partial\bar{V}}{\partial\bar{x}}+\vec{V}\frac{\partial^2\bar{V}}{\partial\bar{x}\partial\bar{t}}\right). \qquad (18)$$

Equation (18) contains a single dimensionless parameter

$$\theta=\frac{L^2}{a^2T^2}.$$

The parameter θ indicates how significant nonlinear effects are in this problem.

2.2. Self-similar equation of shock wave propagation and its solution

As it was already mentioned in the introduction, starting with the work of Riemann [9], then Joukowski [7] and Alliveti [8], it is convenient and logical from a physical point of view to represent the solutions of equation (18) in the form of a propagating wave:

$$\bar{V}(\bar{x},\bar{t})=f(\bar{x}-\bar{a}\bar{t}+C)=f(\eta). \qquad (19)$$

Let's go from function $\bar{V}$ to function $f(\eta)$ taking into account the ratio, substituting it into equation (18). In this way we obtain:

$$\frac{d^2f}{d\eta^2}=\frac{d^2f}{d\eta^2}+\theta\left(\frac{df}{d\eta}(-\bar{a})\frac{df}{d\eta}+f\cdot(-\bar{a})\frac{d^2f}{d\eta^2}\right). \qquad (20)$$

For ease of solution obtaining, let's rewrite equation (20) in the following form:

$$\left(1-\bar{a}^2+\bar{a}\theta\cdot f\right)\frac{d^2f}{d\eta^2}=-\bar{a}\theta\left(\frac{df}{d\eta}\right)^2. \qquad (21)$$

It should be noted, taking into account the dimensionless relations, the value $\bar{a}=1$. So, equation (14) is essentially equivalent to a simpler one:

$$f\frac{d^2f}{d\eta^2}=-\left(\frac{df}{d\eta}\right)^2. \qquad (22)$$

In addition to the elementary solution in the form of a constant, equation (22) also has the following two general solutions:

$$f_1(\eta)=\sqrt{2C_1\eta+2C_2}, \quad f_2(\eta)=-\sqrt{2C_1\eta+2C_2}. \qquad (23)$$

Solutions (23), according to (19), indicate two opposite speeds of propagation. If we take as the scale of the domain of propagation, at each moment of time, the shock wave as L_{sw}, then under the conditions

$$f_1(\eta=0)=-f_2(\eta=0)=1,$$
$$f_1(\eta=L_{sw}/2)=f_2(\eta=-L_{sw}/2)=0$$

we obtain

$$f_1(\eta)=\sqrt{1-2\eta/L_{sw}}, \quad 0\le\eta\le L_{sw}/2;$$
$$f_2(\eta)=-\sqrt{1+2\eta/L_{sw}}, \; -L_{sw}/2\le\eta\le 0. \qquad (24)$$

Outside the domain, $-L_{sw}/2\le\eta\le L_{sw}/2$ both solutions (24) converge to zero (see Fig.1), a constant value that is also the solution of equation (22). It should be noted that for fluid mechanics, if the effect of viscosity is neglected, the following results are quite possible: the wave front (direct and inverted) is clearly defined. As already mentioned in the introduction, these solutions are similar to those given in [7] and are observed at the initial stages of shock momentum rising in gases [16].

If you follow the ideas presented in [4], you can use a different approach to obtaining a self-similar equation and its solution. For this purpose, let's at once substitute in the system of equations (9) and (10) the representation in the form:

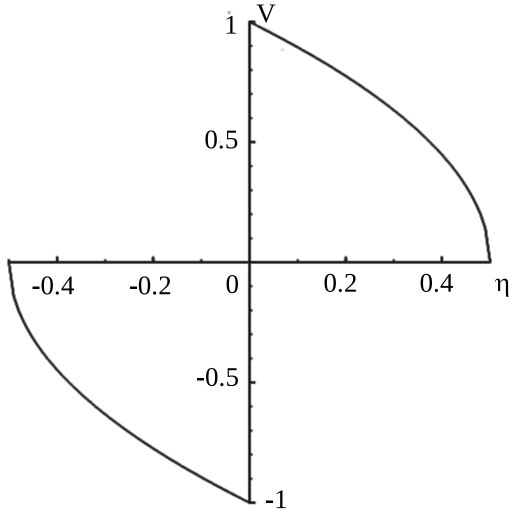

Fig. 1. The velocity distribution function in the shock wave according to solutions (24)

$$\bar{V}=\bar{V}(\eta),\ \bar{p}=\bar{p}(\eta). \qquad (25)$$

Then the dimensionless analogue of the system (9) and (10) takes the following form:

$$-\frac{d\bar{p}}{d\eta}=-\bar{a}\frac{d\bar{V}}{d\eta}+\theta\bar{V}\frac{d\bar{V}}{d\eta}; \qquad (26)$$

$$\bar{a}\frac{d\bar{p}}{dt}=\frac{d\bar{V}}{d\eta}. \qquad (27)$$

Only now we eliminate the pressure function. As a result, we obtain:

$$-\frac{1}{\bar{a}}\frac{d\bar{V}}{d\eta}=(-\bar{a})\frac{d\bar{V}}{d\eta}+\theta\cdot\bar{V}\frac{d\bar{V}}{d\eta}. \qquad (28)$$

For the physical analysis of equation (28), let’s rewrite it in the form:

$$\left(-\frac{1}{\bar{a}}+\bar{a}-\theta\cdot\bar{V}\right)\frac{d\bar{V}}{d\eta}=0. \qquad (29)$$

For the physical analysis of equation (28), let’s rewrite it in the form:

$$\left(-\frac{1}{\bar{a}}+\bar{a}-\theta\cdot\bar{V}\right)\frac{d\bar{V}}{d\eta}=0. \qquad (29)$$

The solutions of (22) are the set of constant numbers. But under condition $\bar{a}=1$ we obtain:

$$\frac{d\left(\bar{V}^2/2\right)}{d\eta}=0. \qquad (30)$$

So, under the condition of approximate constancy of density, we have the fact that the shock wave (pressure) propagates at a constant speed. As we can see, this approach is less meaningful than the previous one: we did not get solutions of the type (24) that differ from the constant.

2.3. Analytical solution in pressure-velocity variables

In classical studies of shock waves, one can find such a concept as the Rankine-Hugonio shock adiabat [17, 18]. As it turned out, in the case of shock wave in a droplet liquid, it is possible to obtain the function

$$\bar{p}=\bar{p}(\bar{V}). \qquad (31)$$

In relation (31), the notation in dimensionless quantities is already used. First, let's write down the dimensionless analogue of system (9) and (10). We obtain:

$$\frac{\partial\bar{V}}{\partial\bar{t}}=-\frac{\partial\bar{p}}{\partial\bar{x}}-\theta\bar{V}\frac{\partial\bar{V}}{\partial\bar{x}}, \qquad (32)$$

$$\frac{\partial\bar{p}}{\partial\bar{t}}=-\frac{\partial\bar{V}}{\partial\bar{x}}. \qquad (33)$$

Let's divide the left and right parts of equation (25) by left and right parts of equation (26), respectively:

$$\frac{\partial\bar{V}}{\partial\bar{p}}=\frac{\partial\bar{p}}{\partial\bar{x}}/\frac{\partial\bar{V}}{\partial\bar{x}}+\theta\bar{V}=\frac{\partial\bar{p}}{\partial\bar{V}}+\theta\bar{V}. \qquad (34)$$

Since we assumed that the pressure depends only on the velocity (the only independent variable), equation (34) can thus be presented in the following form:

$$\left(\frac{d\bar{p}}{d\bar{V}}\right)^2+\theta\bar{V}\frac{d\bar{p}}{d\bar{V}}-1=0. \qquad (35)$$

Equation (35) has two solutions:

$$\frac{d\bar{p}}{d\bar{V}}=\frac{-\theta V\pm\sqrt{(\theta V)^2+4}}{2}. \qquad (36)$$

The solutions of equations (36) are such the functions of pressure versus velocity:

$$\bar{p}_1\left(\bar{V}\right)=\frac{-\bar{V}\sqrt{\left(\theta\bar{V}\right)^2+4}}{4}-$$
$$-\frac{1}{\sqrt{\theta^2}}\ln\left(\frac{\theta^2\bar{V}}{\sqrt{\theta^2}}+\sqrt{\left(\theta\bar{V}\right)^2+4}\right)-$$
$$-\frac{\theta\bar{V}^2}{4}+C_1.$$

$$\bar{p}_2\left(\bar{V}\right)=\frac{\bar{V}\sqrt{\left(\theta\bar{V}\right)^2+4}}{4}+$$
$$+\frac{1}{\sqrt{\theta^2}}\ln\left(\frac{\theta^2\bar{V}}{\sqrt{\theta^2}}+\sqrt{\left(\theta\bar{V}\right)^2+4}\right)-$$
$$-\frac{\theta\bar{V}^2}{4}+C_1.$$

The constant C_1 is determined from the condition:

$$\bar{p}\left(\bar{V}=0\right)=\bar{p}_0. \qquad (37)$$

In equation (37) $\bar{p}_0$ means undisturbed pressure. Since we do not know the exact form of the pressure in the given solutions, it should be found. For this we will write down

$$\frac{\partial\bar{p}}{\partial\bar{x}}=\frac{\partial\bar{p}}{\partial\bar{V}}\frac{\partial\bar{V}}{\partial\bar{x}}. \qquad (38)$$

Now let's rewrite the equation (32) in such the form:

$$-\left(\frac{\partial\bar{V}}{\partial\bar{t}}+\theta\bar{V}\frac{\partial\bar{V}}{\partial\bar{x}}\right)=\frac{\partial\bar{p}}{\partial\bar{x}}=\frac{d\bar{p}}{d\bar{V}}\frac{\partial\bar{V}}{\partial\bar{x}}. \qquad (39)$$

Next, it is convenient to switch to the self-similar variable in equation (39). We have:

$$-\left(-a\frac{d\bar{V}}{d\eta}+\theta\bar{V}\frac{d\bar{V}}{d\eta}\right)=\frac{d\bar{p}}{d\bar{V}}\frac{d\bar{V}}{d\eta}. \qquad (40)$$

Substitution of expression (36) into equation (40) results into:

$$a\frac{d\bar{V}}{d\eta}-\theta\bar{V}\frac{d\bar{V}}{d\eta}=\frac{-\theta V\pm\sqrt{\left(\theta V\right)^2+4}}{2}\frac{d\bar{V}}{d\eta}. \qquad (41)$$

Equation (41) not only indicates the constant velocity of propagation of the shock wave, but also allows us to find the value of this speed. Indeed, after reducing to the derivative, we have

$$\bar{V}=\frac{a^2-1}{a\theta}. \qquad (42)$$

Now the problem can be treated to be completely solved: it was found that under the conditions of the absence (not taking into account) of friction, the shock wave propagates at a constant speed, and the value of the pressure in the liquid can be found by analytical function.

3. Nonlinear models of water hammer in which frictional forces are taken into account

3.1. Taking into account the viscous mechanism of the momentum exchange due to the spatial variability of the flow

If the friction on the wall is not taken into account, even in the hydraulic approximation, then the viscosity is taken into account by the fact that the Navier-Stokes equations are used instead of the Euler equations. So, instead of equation (32), we have a dimensionless equation

$$\frac{\partial\bar{V}}{\partial\bar{t}}+\theta\bar{V}\frac{\partial\bar{V}}{\partial\bar{x}}=-\frac{\partial\bar{p}}{\partial\bar{x}}+\frac{1}{Re}\frac{\partial^2\bar{V}}{\partial\bar{x}^2}. \qquad (43)$$

In equation (43) $Re=[V][L]/\nu$ is Reynolds number. This equation physically corresponds to the propagation of the wave taking into account its viscous blur. In self-similar variables, the system of equations (27), (43) has the form of equation (28) and the following:

$$-a\frac{\partial\bar{V}}{\partial\eta}+\theta\bar{V}\frac{\partial\bar{V}}{\partial\eta}=-\frac{1}{a}\frac{\partial\bar{V}}{\partial\eta}+\frac{1}{Re}\frac{\partial^2\bar{V}}{\partial\eta^2},$$

or, in a form convenient for integration

$$\frac{\partial^2\bar{V}}{\partial\eta^2}=Re\left(\theta\bar{V}-a+\frac{1}{a}\right)\frac{\partial\bar{V}}{\partial\eta}. \qquad (44)$$

Equation (44) has three solutions. The first is a constant speed value, the second and third are like these

$$\bar{V}(\eta)=\frac{a^2-1}{a\theta}\pm\frac{\sqrt{2}\sqrt{C_1}}{\sqrt{Re\theta}}\times$$
$$\times tg\left[\sqrt{2Re\theta C_1}\left(C_2+\eta\right)\right].$$

Qualitative analysis of the obtained solution indicates a propagating wave, and inside it only near the points for which

$$\sqrt{2\,\mathrm{Re}\,\theta C_1}\left(C_2+\eta\right)\approx\pm\frac{\pi}{2},$$

the velocity amplitude is significantly different from the constant. This relatively thin layer directly at the front of the shock wave is called, by analogy with the thin boundary layer, the shock layer.

3.2. A model that takes into account both friction on the walls and viscous exchange of momentum, convection as well

According to the title, the momentum conservation equation now includes the longitudinal gradient (the Weisbach-Darcy model [12, 13]) of the velocity and the inhomogeneity of the velocity field along the flow direction (pipe axis):

$$\rho_0\left(\frac{\partial V}{\partial t}-V\frac{\partial V}{\partial x}+\frac{\lambda}{8R}V\left|V\right|\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 V}{\partial x^2}. \quad (45)$$

The mass conservation equation remains the same that is (10). Dimensional approach more convenient now, since we still need to estimate the ratio of forces. Instead of the term with the speed modulus, you can write a double sign

$$V\left|V\right|=\begin{cases}-V^2, & V<0;\\ V^2, & V>0.\end{cases} \quad (46)$$

Now let's use self-similar variable. In this way:

$$\rho_0\left(-a\frac{dV}{d\eta}-V\frac{dV}{d\eta}\pm\frac{\lambda}{8R}V^2\right)=-\frac{dp}{d\eta}+\mu\frac{d^2V}{d\eta^2}, \quad (47)$$

$$a\frac{dp}{d\eta}=a^2\rho_0\frac{dV}{d\eta}. \quad (48)$$

Substituting equation (48) into (47) gives us:

$$\rho_0\left(-a\frac{dV}{d\eta}-V\frac{dV}{d\eta}\pm\frac{\lambda}{8R}V^2\right)=-a\rho_0\frac{dp}{d\eta}+\mu\frac{d^2V}{d\eta^2}. \quad (49)$$

After simplification, equation (49) becomes:

$$\nu\frac{d^2V}{d\eta^2}=V\frac{dV}{d\eta}\pm\frac{\lambda}{8R}V^2. \quad (50)$$

In equation (50) ν is kinematic viscosity. Despite the apparently relatively simple form of equation (50), its solution is quite complex and is not presented in an explicit form. Since the value of the coefficient of molecular viscosity is significantly less than unity and both nonlinear terms are practically everywhere except for a thin layer with a sharp continuous velocity gradient, it is logical to introduce the ratio of molecular viscosity to the viscosity on the pipe walls and rewrite equation (50) in a more convenient for analysis form:

$$\varepsilon\frac{d^2V}{d\eta^2}=V\left(\frac{8R}{\lambda}\frac{dV}{d\eta}\pm V\right), \quad (51)$$

where $\varepsilon=\nu 8R/\lambda$ is a small parameter, the physical meaning of which is the ratio of the molecular viscosity to the friction force on the pipe wall.

Let's apply the theory of asymptotic approximations with a small parameter to equation (51). According to this theory, we present the unknown velocity function in the form of a power series with on small parameter:

$$V(\eta)=V_0(\eta)+\varepsilon V_1(\eta)+\varepsilon^2 V_2(\eta)+\ldots \quad (52)$$

The corresponding approximations, $V_0(\eta), V_1(\eta), V_2(\eta)\ldots$, are found by substituting (52) into (51) and picking up all terms with the corresponding powers ε. For the "0-th" approximation, the following equation is obtained:

$$0=V_0\left(\frac{8R}{\lambda}\frac{dV_0}{d\eta}\pm V_0\right). \quad (53)$$

The solution of (50) is

$$V_0(\eta)=C_1\exp\left(\mp\frac{\lambda}{8R}\eta\right). \quad (54)$$

Recall that the signs in the argument are chosen in such a way that the exponent decreases. That is, "-" corresponds to positive values of the self-similar variable, and "+" – to negative values. Fig. 2 shows a graph of this function.

The next approximation is found from the following equation

$$\varepsilon\left(\frac{d^2V_0}{d\eta^2}+\varepsilon\frac{d^2V_1}{d\eta^2}\right)=\left(V_0+\varepsilon V_1\right)\times$$
$$\times\left(\frac{8R}{\lambda}\left(\frac{dV_0}{d\eta}+\varepsilon\frac{dV_1}{d\eta}\right)\pm\left(V_0+\varepsilon V_1\right)\right). \quad (55)$$

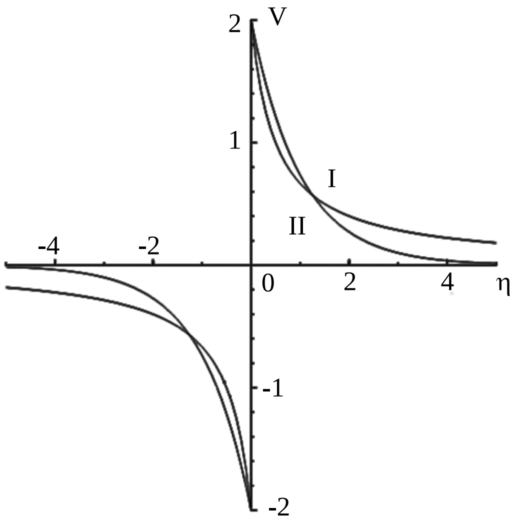

Fig. 2. Velocity distribution: curve I corresponds to taking into account only friction on the walls (8); curve II also takes into account convection (54)

We pick up the terms with the first power of the small parameter ε. The following equation is obtained:

$$\frac{d^2V_0}{d\eta^2}=\frac{8R}{\lambda}\left(V_0\frac{dV_1}{d\eta}+V_1\frac{dV_0}{d\eta}\right)\pm 2V_0V_1. \qquad (56)$$

Now let's use the explicit form of the 0-th order approximation (54) and substitute it into the equation (55). After algebraic simplifications, the following equation is obtained:

$$\frac{dV_1}{d\eta}=\mp\frac{\lambda}{8R}V_1+\left(\frac{\lambda}{8R}\right)^3. \qquad (57)$$

The solution (57) has the following form:

$$V_1(\eta)=\left(\frac{\lambda}{8R}\right)^3\eta+C_2\exp\left(\mp\frac{\lambda}{8R}\eta\right). \qquad (58)$$

The first term in (58) is a linear function of the self-similar variable. Therefore, in order to be able to "bend" the solution and force it to approach zero, we take into account one more approximation - the second order of smallness with ε. We write the required equation again in the following form:

$$\varepsilon\left(\frac{d^2V_0}{d\eta^2}+\varepsilon\frac{d^2V_1}{d\eta^2}+\varepsilon^2\frac{d^2V_2}{d\eta^2}\right)=\left(V_0+\varepsilon V_1+\varepsilon^2V_2\right)\times$$
$$\times\left(\frac{8R}{\lambda}\left(\frac{dV_0}{d\eta}+\varepsilon\frac{dV_1}{d\eta}+\varepsilon^2\frac{dV_2^2}{d\eta^2}\right)\pm\left(V_0+\varepsilon V_1+\varepsilon^2V_2\right)\right).$$

After algebraic transformations and reductions that take into account the explicit form of previous approximations, the following equation is obtained:

$$\frac{dV_2}{d\eta}\pm DV_2=-\frac{D^6}{C_1}\eta\exp(\pm D\eta). \qquad (59)$$

where $D=\lambda/8R$,

$$V_2(\eta)=\left(-\frac{\exp((D+1)\eta)D^6((D+1)\eta-1)}{C_1(D+1)^2}+C_2\right)\exp(-\eta),$$
$$\eta\ge 0.$$

$$V_2(\eta)=\left(\frac{\exp(-(D+1)\eta)D^6((D+1)\eta+1)}{C_1(D+1)^2}+C_2\right)\exp(\eta),$$
$$\eta\le 0.$$

So, the general form of the asymptotic solution with accuracy up to the second order on a small parameter ε is obtained. As the quantitative analysis of taking into account the first and second asymptotic approximations for smooth pipes showed, they are negligibly small compared to the zero approximation. Therefore, for applied calculations it is sufficient to limit ourselves to expression (54). On the other hand, comparing the curves in Fig. 2 shows the importance of convection velocity fields during shock wave propagation.

Discussion

Undoubtedly, in the modern world, the computer experiment plays a significant role in the research of complex phenomena, in particular, hydraulic shock. Today, there are not only one-dimensional models of shock wave propagation in droplet liquid, but also two-dimensional ones, as well as those that take into account the turbulent nature of the flow during the passage of the shock wave [19]. But, on the other hand, the rapid development of computer technology made it possible the analytical solution finding in for many nonlinear differential equations and their systems. The combination of the experience of obtaining self-similar equations and their solutions opened up new opportunities for finding analytical solutions that describe the phenomenon of hydraulic shock. For technical needs, this is absolutely necessary, as it makes it possible to estimate the pressure increase and the consequences of this increase - elastic deformation of the pipeline surface and fatigue phenomena [20]. These deformations are associated with the translational motion of cavitation accompanying the shock wave [21]. Ultimately, they lead to

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fatigue corrosion of the pipe surface [22], which is an extremely undesirable phenomenon.

Conclusions

The paper considers the phenomenon of unsteady flow in a droplet liquid. An unsteady flow essentially forms a shock momentum, although it is commonly called a shock wave. Based on the previous works of the authors, and taking into account the achievements of others, an analytical study of nonlinear and viscous effects during shock wave propagation was carried out. Molecular viscosity, according to obtained solution, is significant only in the thin shock layer, at the wave front. But the convection mechanism of the momentum (velocity field) should be taken into account in the entire domain of the shock wave's existence. As it turned out, an analytical relationship between the pressure function and the velocity function can be found for a droplet liquid within the framework of the inviscid approximation.

Future research could consider unsteady flow in a multiphase fluid.

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Conflict of Interest

The authors declare that they have no conflict of interest in relation to this research, whether financial, personal, authorship or otherwise, that could affect the research and its results presented in this paper.

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Data availability

The manuscript has no associated data.

Use of Artificial Intelligence

The authors confirm that they did not use artificial intelligence methods while creating the presented work.

All the authors have read and agreed to the published version of this manuscript.

References

1. Krashanitsa, Yu. A., & Zheryakov, D. Yu. Aerodinamicheskiy profil' v transzvukovom potoke gaza [Airfoil section in the near-sonic flow of gas]. *Aviacijno-kosmicna tehnika i tehnologia – Aerospace technic and technology*, 2021, no. 2(170), pp. 20-27. DOI: 10.32620/aktt.2021.2.03. (in Russian).

2. Khorokhordin, A. O., Kravchenko, I. F., Mitrokhovych, M. M., Balalayeva, K. V., Balalayev, A. V. Metodyka ratsionalʹnoho formuvannya poverkhonʹ halʹmuvannya ploskoho nadzvukovoho vkhidnoho prystroyu [Method of rational formation of braking surfaces of a flat supersonic entry]. *Aviacijno-kosmicna tehnika i tehnologia – Aerospace technic and technology*, 2023, no. 4(190), pp. 28-34. DOI: 10.32620/aktt.2023.4sup2.03. (in Ukrainian).

3. Sivashenko, T. I. *Dinamicheskiye i prochnostnyye kharakteristiki gibkikh metallicheskikh i ftoroplastovykh truboprovodov sistem letatel'nykh apparatov*. Dis. ... kand. tekhn. nauk. [Dynamic and strength characteristics of flexible metal and fluoroplastic pipelines of aircraft system. Diss. … cand. tech. sci.]. Kyiv, 1972. 181 p. (in Russian).

4. Lukianov, P. V., Syvashenko, T. I., & Yakymenko, B. M. Udarna khvylya u ridyni, shcho znakhodytʹsya u pruzhniy tsylindrychniy obolontsi neskinchenoyi dovzhyny [Shock wave in a liquid located in an elastic cylindrical shell of infinite length]. *Promyslova hidravlika i pnevmatyka – Industrial hydraulics and pneumatics*, 2019, vol. 2(64), pp. 38-46. (in Ukrainian).

5. Menabrea, L. F. Note sur les effects de choc de l'eau dans les conduits. *C.R. Hebd. Seances Acad. Sci.*, Paris, Impr. de Mallet-Bachelier Publ., 1885, vol. 47, pp. 221-224. (in French).

6. Michaud, J. Coup de belier dans les conduits. Etude des moyens employes pour en atteneur less effects. *Bulletin de la Société vaudoise des ingénieurs et des architectes*, 1878, vol. 4(3.4), pp. 56-64, 65-77. DOI: 10.5169/seals-5907. (in French).

7. Joukowski, N. E. Memories of Imperial Academy Society of St. Petrburg, 9(5). (Russian translated by O. Simin 1904) *Proc. Amer. Water Assoc.*, 1898, vol. 24, pp. 341-424.

8. Allievi, L. Teoria generale del moto perturbato dell'acqua nei tubi in pressione (colpo d'ariete). *Ann. Soc. Ing. Arch. Ithaliana*, 1903, vol. 17, pp. 285-325. (in Italian).

9. Riemann, B. *Ueber die Fortpflanzung Ebener Luftwellen von Endlicher Schwingungsweite (German Edition).* Leopold Classic Library Publ., 2017. 32 p. (In German).

10. Ghidaoui, M. S., Zhau, M., McInnis, D. A., & Axworthy, D. H. A review of Water Hummer Theory and Practice. *Applied Mechanics Reviews*, 2005, vol. 58, iss. 1, pp. 49-76. DOI: 10.1115/1.1828050.

11. Kolpak, E. P., & Ivanov, S. I. Mathematical and Computer Modeling Vibrstion Protection System with Damper. *Applied Mathematical Science*, 2015, vol. 9, no. 78, pp. 3875-3885. DOI: 10.12988/ams.2015.53270.

12. Darcy, H. *Recherches expérimentales relatives au mouvement de l'eau dans les tuyaux*. Mallet-Bachelier, Paris, 1857. 268 p. Available at: https://archive.org/details/bub_gb_a4rcE52LFF0C/page/n11/mode/2up (accessed 10 Dec. 2023) (in French).

13. Weisbach, J. *Lehrbuch der Ingenieur- und Maschinen-Mechanik, Vol. 1. Theoretische Mechanik*, Braunschweig, Vieweg Publ., 1855. 946 p. Available at: https://www.digitale-sammlungen.de/de/view/bsb10081227 (accessed 10 Dec. 2023) (In German).

14. Tijsseling, A. S., Lambert, M. F., Simpson, A. R., Stephens, M. L., Vitkovsky, J. P., Bergand, & A. Skalak's extentended theory of water hammer. *Journal of Sound and Vibration*, 2008, vol. 310, iss. 3, pp. 718-728. DOI: 10.1016/j.jsv.2007.10.037.

15. Skalak, R. An extention of the Theory of water hummer. *Trans. ASME*, 1956, vol. 78, no. 1, pp. 105-115. DOI: 10.1115/1.4013579.

16. Soloukhin, R. I. *Udarnyye volny i detonatsiya v gazakh* [Shock waves and detonation in gases]. Moscow, Fizmatlit Publ., 1963. 176 p. (in Russian).

17. Rankine, W. J. M. On the thermodynamic theory of waves of finite longitudinal disturbances. *Philosophical Transactions of Royal Society of London*. 1870, no. 160, pp. 277-288. Available at: https://www.jstor.org/stable/109061 (accessed 10 Dec. 2023).

18. Hugoniot, H. Memoir on the propagation of movements in bodies, especially perfect gases (first part). *Journal de l'Ecole Polytechnique*, 1887, no. 57, pp. 3-97. Available at: https://books.google.com.ua/books?id=wccAAAAAYAAJ&pg=PA3&redir_esc=y#v=onepage&q&f=false (accessed 10 Dec. 2023) (in French).

19. Su, H., Sheng, L., Zhao, S., Lu, C., Zhu, R., Chen, Y., & Fu, Q. Water Hammer Characteristics and Component Fatigue Analysis of the Essential Service Water System in Nuclear Power Plants. *Processers*, 2023, vol. 11, iss. 12, article no. 3305. DOI: 10.3390/pr11123305.

20. Adamkowski, A., Lewandowsky, M., & Lewandowsky, S. Fatigue life analysis of hydropower pipelines using the analytical model of stress concentration in welded joints with angular distortions and considering the influence of water hammer damping. *Thin-Walled Structures*, 2021, vol. 159, article no. 107350, pp. 1-12. DOI: 10.1016/j.tws.2020.107350.

21. Walters, T. W., & Leishear, R. A. When the Joukowsky Equation Does Not Predict Maximum Water Hammer Pressure. *Journal of Pressure Vessel Technology*, 2019, vol. 141, iss. 6, article no. 060801. 10 p. DOI: 10.1115/1.4044603.

22. Leishear, R. A. Water hammer and Fatigue Corrosion –I –A Piping System Failure Analysis. *Leishear Engineering, LLC*, 2023, pp. 1-19. DOI: 10.13140/RG.2.2.19541.09449.

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НЕСТАЦІОНАРНА ТЕЧІЯ КРАПЕЛЬНОЇ РІДИНИ В ГІДРАВЛІЧНИХ СИСТЕМАХ ЛІТАКІВ І ВЕРТОЛЬОТІВ: МОДЕЛІ ТА АНАЛІТИЧНІ РОЗВ'ЯЗКИ

П. В. Лук'янов, К. С. Павлова

Предметом даної роботи є нестаціонарна течія рідини в трубопроводах, що є частиною конструкції літаків та вертольотів. Під такою назвою розуміється, перше за все, явище різкого підвищення тиску в трубопроводі, яке відоме як гідравлічний удар. Хоча вже навчилися боротися з цим явищем в окремих частинах систем, але у багатьох елементах конструкцій (гнучкі трубопроводи), всередині яких робочій тиск сягає кількохсот атмосфер, це явище ще й досі є доволі небезпечним. Найкращій спосіб боротьби із небажаним явищем, як відомо, є його теоретичне вивчення. На сьогоднішній день наявна величезна кількість робіт в напрямку досліджень гідравлічного удару. Дана стаття не ставить за мету повне охоплення цих робіт. В ній обмежується посиланням на оглядові та відповідні роботи. Оскільки явище гідравлічного удару має суттєво нелінійний характер, то донедавна були невідомі аналітичні розв'язки систем рівнянь, що відповідають найпростішим моделям. У даній роботі представлено, у якості огляду, вже відомі аналітичні розв'язки, що описують процес поширення ударної хвилі, і , головне, наведено нові здобутки, як для нев'язкого наближення, так і з урахуванням внутрішнього в'язкого тертя. Показано, що внутрішнє тертя, в рамках розглянутої моделі, є незначним майже усюди, окрім тонкого ударного шару. Асимптотика пропорційна функції тангенса і обернено пропорційна кореню квадратному із добутку числа Рейнольдса та безрозмірного параметра,

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що характеризує ефект конвекції. В той же час конвекція поля швидкості істотно впливає на розподіли характеристик у гідравлічному ударі. Якщо автомодельні розв'язки, що були отримані раніше, мають степеневий характер розподілу швидкості в ударній хвилі, то одночасне врахування в моделі конвекції і тертя на стінках трубопроводу (за моделлю Вейсбаха-Дарсі) дало змогу отримати розподіл у вигляді експоненціальної функції, що спадає зі зростанням відстані від фронту ударної хвилі. Крім того, в роботі здійснено оригінальний підхід щодо розв'язання нелінійної системи диференціальних рівнянь, що описує поширення ударної хвилі без урахування тертя на стінках. Отримані аналітичні розв'язки у вигляді функції тиску від швидкості поширення ударної хвилі. **Методи досліджень.** В роботі використовуються суто теоретичні підходи, що базуються на використанні відомих моделей течії крапельної рідини, методах аналітичного розв'язання диференціальних рівнянь та їх систем, асимптотичні методи, виведення автомодельних рівнянь та знаходження їх розв'язків. **Висновки.** Отримана аналітичні розв'язки систем диференціальних рівнянь, які описують моделі гідравлічного удару як без урахування в'язких ефектів , так із урахуванням їх. Наведено порівняння отриманих результатів із результатами інших робіт.

Ключові слова: літак; вертоліт; елементи конструкції; гідравлічний удар (ударна хвиля); напруження; тертя; деформація поверхні; втома.

Лук'янов Павло Володимирович – канд. фіз.-мат. наук, старш. наук. співроб., доц. каф. гідрогазових систем, Національний авіаційний університет, Київ, Україна.

Павлова Катерина Сергіївна – магістр, асп. каф. гідрогазових систем, Національний авіаційний університет, Київ, Україна.

Pavlo Lukianov – Candidate of Physics and Mathematics Sciences, Senior Researcher, Associate Professor of Hydro-Gas Systems Department, National Aviation University, Kyiv, Ukraine,
e-mail: Pavlo.Lukianov@npp.nau.edu.ua, ORCID: 0000-0002-5043-6182.

Kateryna Pavlova – Master of Technical Sciences, PhD Student of Hydro-Gas Systems Department, National Aviation University, Kyiv, Ukraine,
e-mail: pavlovadazv@gmail.com, ORCID: 0000-0001-8818-4358.