UDC 517.5

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ATOMIC WAVELETS

The problem of existence and construction of the atomic wavelet system, which consists of infinitely differentiable functions with a compact support, is considered. Formulas for evaluation atomic wavelets are obtained. Examples of applications of atomic wavelets to approximation of some functions are presented. Compactly supported solutions of some functional differential equations and their properties are considered. A new class of atomic functions is introduced. Approximation properties of the linear spaces of finite linear combinations of translates of the atomic functions are presented.

Key words: *wavelet, functional differential equation, atomic function, atomic wavelet, function with compact support, infinitely differentiable function, nonstationary wavelet system.*

Introduction

The theory of wavelets is an intensively developing branch of pure and applied mathematics. Methods of wavelet analysis are used in astronomy and astrophysics [1, 2], computer graphics [3-5], digital signal processing [6-9], economics [10, 11], geophysics [12, 13], medicine [14] and so on. There are many monographs on mathematical aspects of the wavelet theory [15-18].

There are many ways of defining a wavelet. Generally, the function $\varphi \in L_2(R)$ is a wavelet, if the system of functions

$$\Omega = \left\{ 2^{\frac{k}{2}} \cdot \varphi\left(2^k \cdot x - j\right) \right\}_{k,j \in Z}$$

is an orthonormal basis of $L_2(R)$ [19]. The system Ω is constructed using translates of one function. In this case we shall say that Ω is a stationary wavelet system. Let us remark that there is no stationary wavelet system, which consists of infinitely differentiable functions with compact support [20].

In this paper we consider the problem of existence and construction of the atomic wavelet systems that consist of infinitely differentiable compactly supported functions.

1. Formulation of the problem

Consider the functions

$$f_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \cdot F_n(t)dt, \quad g_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \cdot G_n(t)dt,$$

where

$$F_n(t) = \left(\frac{\sin \frac{2t}{4^{n+1}}}{\frac{2t}{4^{n+1}}} \right)^n \cdot F\left(\frac{t}{4^n}\right),$$

$$G_n(t) = \left(\frac{\sin \frac{t}{4^{n+1}}}{\frac{t}{4^{n+1}}} \right)^{n+1} \cdot F\left(\frac{t}{4^{n+1}}\right),$$

$$F(t) = \prod_{k=1}^{\infty} \frac{\sin \frac{2t}{4^k}}{\frac{2t}{4^k}} \cdot \cos \frac{t}{4^k} \text{ and } n = 0,1,2,\ldots.$$

Let $v_{2k}(x) = f_k\left(x - \frac{k+2}{2 \cdot 4^k}\right)$, $v_{2k+1}(x) = g_k\left(x - \frac{k+2}{4^{k+1}}\right)$ for any $k = 0,1,2,\ldots$. These functions are infinitely differentiable [21].

By V_n denote the linear space of functions $\varphi(x)$ such that $\varphi(x) = \sum_{k \in I(\varphi)} c_k(\varphi) \cdot v_n\left(x - \frac{k}{2^n}\right)$, where $I(\varphi)$ is a finite subset of integers. It was proved in [21] that $V_0 \subset V_1 \subset \ldots \subset V_n \subset \ldots$.

In this paper we define the inner product of two functions $f, g \in L_2(R)$ as the integral $\int_R f(x) \cdot g(x)dx$.

Let W_n be the space of functions $\varphi \in V_n$ such that $\varphi \perp W_n$.

Theorem 1 [21]. For any natural n there exists the function $w_n(x)$ such that

1) the system of functions $\left\{ w_n\left(x - \frac{j}{2^{n-1}}\right) \right\}_{j \in Z}$ is a basis of the linear space W_n;

2) $\text{supp } w_n(x) \subseteq \left[0, \frac{n+2}{2^{n-1}}\right]$;

3) for any $m = 0,1,\ldots,\left[\frac{n+1}{2}\right] - 1$

$$\int_R x^m \cdot w_n(x)dx = 0.$$

W_n is a space of wavelets. The system of functions $\Omega = \left\{ v_0(x-j), w_n\left(x - \frac{j}{2^{n-1}}\right) \right\}_{n\in N, j\in Z}$ is a nonstationary wavelet system [18, 22].

Notice that the function $f_0(x)$ is the function $mup_2(x)$, which was introduced by V.A. Rvachev and G.A. Starets in [23]. The function $F(t)$ is a Fourier transform of $mup_2(x)$. Therefore we can say that construction of V_n, W_n and Ω is based on $mup_2(x)$. This function is a solution with compact support of the functional differential equation

$$y'(x) = 2\cdot\left(y(4x+3) - y(4x-1) + y(4x+1) - y(4x-3)\right)$$

and belongs to the class of atomic functions [24, 25]. Thus we shall say that $w_n(x)$ is an atomic wavelet.

Note also that theorem 1 is an existence theorem. This result can not be used in practice for solution of some applied problems.

The aim of this paper is to generalize theorem 1 and obtain formulas for evaluating atomic wavelets.

2. Solution of the problem

2.1. The function $mup_s(x)$ and its properties

Consider atomic function

$$mup_s(x) = \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty} e^{itx}\cdot F_s(t)dt,$$

where $F_s(t) = \prod_{k=1}^{\infty} \frac{\sin^2\left(\frac{s\cdot t}{(2s)^k}\right)}{s^2\cdot\frac{t}{(2s)^k}\cdot\sin\frac{t}{(2s)^k}}$ and $s = 2,3,\ldots$. The function $mup_s(x)$, which is a solution with a compact support of the functional differential equation

$$y'(x) = 2\cdot\sum_{k=1}^{s}\left(y(2s\cdot x + 2s - 2k + 1) - y(2s\cdot x - 2k + 1)\right),$$

was introduced by V.A. Rvachev and G.A. Starets in [23]. Let us remark that this function is a generalization of the function

$$up(x) = \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty} e^{itx}\cdot\prod_{k=1}^{\infty}\frac{\sin\frac{t}{2^k}}{\frac{t}{2^k}}dt,$$

which is a compactly supported solution of the equation $y'(x) = 2\cdot\left(y(2x+1) - y(2x-1)\right)$ [24, 25].

The function $mup_s(x)$ has the following properties:

1) $mup_s(x) \in C^\infty$ [23];

2) $mup_s(x)$ is an even function [23];

3) $\mathrm{supp}\; mup_s(x) = [-1,1]$ [23];

4) $mup_s(0) = 1$ [23];

5) a Fourier transform of $mup_s(x)$ is an entire function of exponential type [24];

6) $mup_s(x) > 0$ for any $x \in (-1,1)$.

To prove the last property we use the formula

$$mup_s\left(-1 + \frac{1}{s\cdot(2s)^n}\right) = c_{s,n}\cdot\int_{-1}^{1}(-\tau+1)^n\cdot mup_s(\tau)d\tau,$$

where $c_{s,n} = \frac{2^{n+1}}{n!\cdot(2s)^{\frac{(n+1)(n+2)}{2}}}$, $s = 2,3,\ldots$ and $n = 0,1,\ldots$ (see lemma 2 in [26]). Let $x_{s,n} = -1 + \frac{1}{s\cdot(2s)^n}$. It follows from properties of $mup_s(x)$ that $mup_s\left(x_{s,n}\right) > 0$. Consider any $x_0 \in (-1,0)$. There exists $n \in N$ such that $x_{s,n} < x_0$. The function $mup_s(x)$ increases on $[-1,0]$ (the proof can be found in [27]). Then $0 < mup_s\left(x_{s,n}\right) \le mup_s\left(x_0\right)$. Combining this with properties 2) and 4) we obtain that $mup_s(x) > 0$ for any $x \in (-1,1)$.

The function $mup_s(x)$ and its Fourier transform will be used to generalize theorem 1.

2.2. Functions $Fmup_{m,n,k}(x)$

Denote by m any natural number.

Consider the functions

$$F_{m,n,0}(t) = \left(\frac{\sin\frac{2^m\cdot t}{2^{(m+1)(n+1)}}}{\frac{2^m\cdot t}{2^{(m+1)(n+1)}}}\right)^n\cdot F_{2^m}\left(\frac{t}{2^{n(m+1)}}\right),$$

$$F_{m,n,k}(t) = \left(\frac{\sin\frac{2^{m-k}\cdot t}{2^{(m+1)(n+1)}}}{\frac{2^{m-k}\cdot t}{2^{(m+1)(n+1)}}}\right)^{n+1}\cdot\prod_{j=0}^{m-1-k}\cos\frac{2^j\cdot t}{2^{(m+1)(n+1)}}\times$$

$$\times F_{2^m}\left(\frac{t}{2^{(n+1)(m+1)}}\right),$$

where $n = 0,1,2,\ldots$, $k = 1,\ldots,m$ and $F_{2^m}(t)$ is a Fourier transform of the function $mup_{2^m}(x)$. These functions are entire functions of exponential type. Moreover, for any $n = 0,1,2,\ldots$ and $k = 0,1,\ldots,m$ the function $F_{m,n,k}(t)$ approaches to zero faster then t^j for any natural j. The function $F_{m,n,k}(t)$ is an atomic function.

Let $Fmup_{m,n,k}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx}\cdot F_{m,n,k}(t)dt$,

where $n = 0,1,2,\ldots$ and $k = 0,1,\ldots,m$. It follows from the Wiener-Paley theorem [28] that $Fmup_{m,n,k}(x) = 0$

for any x such that $|x| > \frac{n+2}{2^{n(m+1)+k+1}}$. Furthermore, these functions are infinitely differentiable.

From the properties of the Fourier transform it follows that

$$\mathrm{Fmup}_{m,n,k}(x) = \frac{1}{2^{n+1}} \cdot \sum_{j=0}^{n+2} \binom{n+2}{j} \times$$

$$\times \mathrm{Fmup}_{m,n,k+1}\left(x + \frac{2^{m-k} \cdot j - 2^{m-k-1} \cdot (n+2)}{2^{(m+1)(n+1)}}\right), \qquad (1)$$

$$\mathrm{Fmup}_{m,n,m}(x) = \frac{1}{2^{n+1}} \cdot \sum_{j=0}^{n+1} \binom{n+1}{j} \times$$

$$\times \mathrm{Fmup}_{m,n+1,0}\left(x + \frac{2^{m+1} \cdot j - 2^{m} \cdot (n+2)}{2^{(m+1)(n+2)}}\right), \qquad (2)$$

where $n = 0,1,2,\ldots$ and $k = 0,1,\ldots,m-1$.

Note that the function $\mathrm{Fmup}_{m,n,k}(x)$ is a generalization of the function $\mathrm{Fup}_n(x)$ [24, 25].

2.3. Atomic wavelets

Let m be a fixed natural number. Consider the function

$$v_{m,n\cdot(m+1)+k}(x) = \mathrm{Fmup}_{m,n,k}\left(x - \frac{n+2}{2^{n\cdot(m+1)+k+1}}\right),$$

where $n = 0,1,2,\ldots$ and $k = 0,1,\ldots,m$.

By $V_{m,n}$ denote the space of functions $\varphi(x)$ such that $\varphi(x) = \sum_{j \in I(\varphi)} c_j(\varphi) \cdot v_{m,n}\left(x - \frac{j}{2^n}\right)$, $x \in R$, where $I(\varphi)$ is a finite subset of integers, $n = 0,1,2,\ldots$

Let $\mathrm{WUP}_{m,n} = \{\varphi \in V_{m,n} : \varphi \perp V_{m,n-1}\}$ for any natural n.

The following result was obtained.

Theorem 2. For any $m = 1,2,\ldots$ and $n = 1,2,\ldots$ there exists the function $\mathrm{wup}_{m,n}(x)$ such that

1) the system of function $\left\{\mathrm{wup}_{m,n}\left(x - \frac{j}{2^{n-1}}\right)\right\}_{j \in Z}$ is a basis of the linear space $\mathrm{WUP}_{m,n}$;

2) $\mathrm{supp}\, \mathrm{wup}_{m,n}(x) \subseteq \left[0, \frac{n+2}{2^{n-1}}\right]$;

3) $\int_R \mathrm{wup}_{m,n}(x)dx = 0$.

We say that the function $\mathrm{wup}_{m,n}(x)$ is **an atomic wavelet**, the space $\mathrm{WUP}_{m,n}$ is **a space of atomic wavelets** and the system of functions

$$\Omega_m = \left\{\mathrm{mup}_{2^m}(x - j),\ \mathrm{wup}_{m,n}\left(x - \frac{j}{2^{n-1}}\right)\right\}_{n \in N, j \in Z}$$

is **a system of atomic wavelets**.

Theorem 2 is a generalization of theorem 1.

2.4. Functions $g_{m,k}(x)$

First note that theorem 2 is only an existence theorem. We need convenient formulas for evaluation atomic wavelets. In this subsection we introduce the functions $g_{m,k}(x)$ that will be used to construct atomic wavelets $w_{m,n}(x)$.

Let $g_{m,k}(x) = \frac{1}{2^k} \cdot \mathrm{Fmup}_{m,0,k}(x)$, $k = 0,1,\ldots,m$.

These function combine the following properties:

$$g_{m,k}(x) \in C^{\infty}, \qquad (3)$$

$$\mathrm{supp}\, g_{m,k}(x) = \left[-\frac{1}{2^k}, \frac{1}{2^k}\right], \qquad (4)$$

$$g_{m,k}(x) = \frac{1}{2} \cdot \left(g_{m,k+1}\left(x - \frac{1}{2^{k+1}}\right) + 2 \cdot g_{m,k+1}(x) + \right.$$

$$\left. + g_{m,k+1}\left(x + \frac{1}{2^{k+1}}\right)\right),\ k = 0,1,\ldots,m-1. \qquad (5)$$

The proof of these properties is trivial.

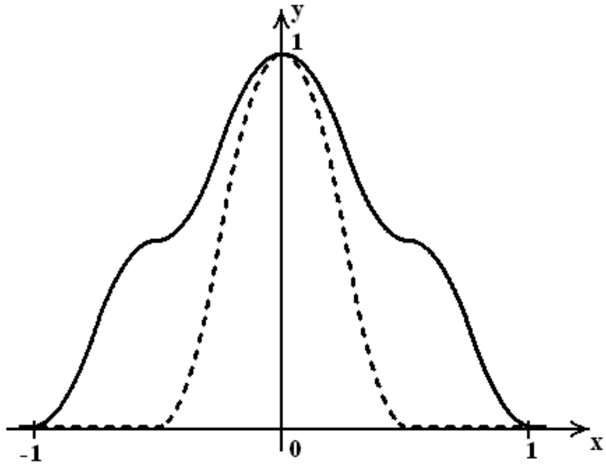


Fig. 1. Graphs of $g_{1,0}(x)$ and $g_{1,1}(x)$:
—— – $g_{1,0}(x)$; ---- – $g_{1,1}(x)$

Let $\widetilde{G}_m$ be the space of functions $\varphi(x)$ such that

$$\varphi(x) = \sum_j c_j \cdot g_{m,m}\left(\frac{x}{\pi} - \frac{j}{2^m}\right),\ x \in [-\pi, \pi]$$

and $\varphi^{(p)}(-\pi) = \varphi^{(p)}(\pi)$ for any $p = 0,1,2,\ldots$. Dimension of $\widetilde{G}_m$ is equal to 2^{m+1}.

By $\widetilde{W}_2^1$ denote a class of functions $f \in C_{[-\pi,\pi]}$ such that $f(-\pi) = f(\pi)$, f(x) is absolutely continuous and $\|f'\|_{L_2[-\pi,\pi]} \le 1$.

The following theorem was obtained.

Theorem 3. For any natural m it is true that

$$E_{L_2}\left(\widetilde{W}_2^1, \widetilde{G}_m\right) \le C \cdot d_{2^{m+1}}\left(\widetilde{W}_2^1, L_2[-\pi, \pi]\right),$$

where $E_X(K,L)=\sup_{\varphi\in K}\inf_{\psi\in L}\|\varphi-\psi\|_X$ is the best approximation of the class K by L in norm of X;

$d_N(K,X)=\inf_{\dim L=N}E_X(K,L)$ is the Kolmogorov width [29].

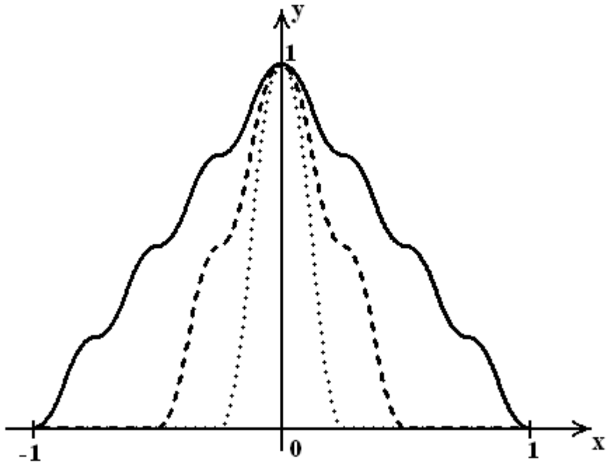


Fig. 2. Graphs of $g_{2,0}(x), g_{2,1}(x)$ and $g_{2,2}(x)$:
—— – $g_{2,0}(x)$; – – – – – $g_{2,1}(x)$; ⋯⋯ – $g_{2,2}(x)$

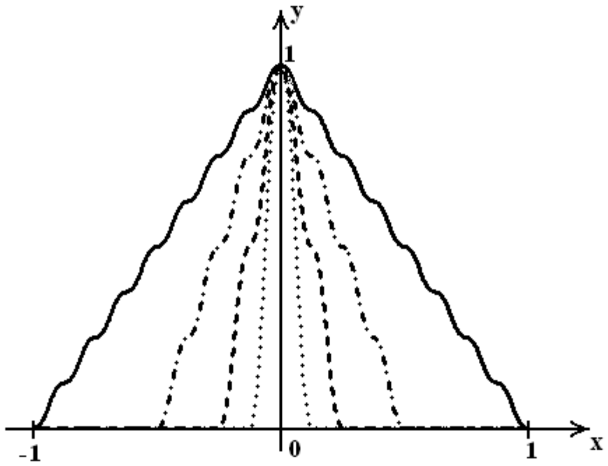


Fig. 3. Graphs of $g_{3,0}(x), g_{3,1}(x), g_{3,2}(x)$ and $g_{3,3}(x)$:
—— – $g_{3,0}(x)$; –·–·– – $g_{3,1}(x)$; – – – – – $g_{3,2}(x)$; ⋯⋯ – $g_{3,3}(x)$

So $g_{m,m}(x)$, which is infinitely differentiable function with a compact support, has good approximation properties.

2.5. Construction of atomic wavelets

First note that for any $m\in N$, $k=0,1,\ldots,m$ and $f\in V_{m,k}$ the following formula is true

$$f(x)=\sum_j c_j(f)\cdot g_{m,k}\left(x-\frac{j}{2^k}\right).$$

Let m and k be fixed numbers such that $m\in N$ and $k=1,\ldots,m$.

Consider the function $\varphi(x)=\sum_{j=1}^{5}x_j\cdot g_{m,k}\left(x-\frac{j}{2^k}\right)$.

From (4) it follows that this function belongs to $WUP_{m,k}$ if and only if it satisfies the condition

$$\varphi(x)\perp g_{m,k-1}\left(x-\frac{p}{2^{k-1}}\right) \qquad (6)$$

for any $p=0,1,2,3$. Therefore a necessary and sufficient condition for $\varphi(x)$ to be an element of the linear space $WUP_{m,k}$ is that coefficients x_j satisfy the system of linear algebraic equations

$$\sum_{j=1}^{5}a_{ij}^{[m,k]}\cdot x_j=0,\ i=0,1,2,3.$$

Using (4) and (5) we get

$$\begin{cases}\left(b_{m,k}+\frac{1}{2}a_{m,k}\right)\cdot x_1+\frac{1}{2}b_{m,k}\cdot x_2=0\\ \left(b_{m,k}+\frac{1}{2}a_{m,k}\right)\cdot x_1+\left(a_{m,k}+b_{m,k}\right)\cdot x_2+\\ \quad+\left(b_{m,k}+\frac{1}{2}a_{m,k}\right)\cdot x_3+\frac{1}{2}b_{m,k}\cdot x_4=0\\ \frac{1}{2}b_{m,k}\cdot x_2+\left(b_{m,k}+\frac{1}{2}a_{m,k}\right)\cdot x_3+\\ \quad+\left(a_{m,k}+b_{m,k}\right)\cdot x_4+\left(b_{m,k}+\frac{1}{2}a_{m,k}\right)\cdot x_5=0\\ \frac{1}{2}b_{m,k}\cdot x_4+\left(b_{m,k}+\frac{1}{2}a_{m,k}\right)\cdot x_5=0\end{cases}$$

where $a_{m,k}=\int_R g_{m,k}^2(x)dx$,

$$b_{m,k}=\int_R g_{m,k}(x)\cdot g_{m,k}\left(x-\frac{1}{2^k}\right)dx \text{ (table 1).}$$

It is easy to solve this system. We get $x_1=\alpha\cdot(-b_{m,k})$, $x_2=\alpha\cdot(a_{m,k}+2\cdot b_{m,k})$, $x_3=\alpha\cdot(-2\cdot a_{m,k}-2\cdot b_{m,k})$, $x_4=\alpha\cdot(a_{m,k}+2\cdot b_{m,k})$ and $x_5=\alpha\cdot(-b_{m,k})$, where α is an arbitrary real number.

Consider the function

$$wup_{m,k}(x)=-b_{m,k}\cdot g_{m,k}\left(x-\frac{1}{2^k}\right)+\left(a_{m,k}+\frac{1}{2}\cdot b_{m,k}\right)\times$$
$$\times g_{m,k}\left(x-\frac{2}{2^k}\right)+(-2)\cdot\left(a_{m,k}+b_{m,k}\right)\cdot g_{m,k}\left(x-\frac{3}{2^k}\right)+$$
$$+\left(a_{m,k}+\frac{1}{2}\cdot b_{m,k}\right)\cdot g_{m,k}\left(x-\frac{4}{2^k}\right)-b_{m,k}\cdot g_{m,k}\left(x-\frac{5}{2^k}\right).$$

It is clear that $wup_{m,k}(x)\in WUP_{m,k}$. Furthermore, the system of functions $\left\{wup_{m,k}\left(x-\frac{j}{2^{k-1}}\right)\right\}_{j\in Z}$ is a basis of the linear space $WUP_{m,k}$. This yields that $wup_{m,k}(x)$ is an atomic wavelet.

Table 1

Values of $a_{m,k}$ and $b_{m,k}$

m	k	$a_{m,k}$	$b_{m,k}$
1	0	0.6916650904158232	0.15417717500562
	1	0.3833333333333334	0.05833333333333
2	0	0.6729161412497189	0.16354192937514
	1	0.3458322824994376	0.07708385875028
	2	0.1916656158327709	0.02916666666667
3	0	0.6682764606040972	0.16586176969795
	1	0.336505495916527	0.0817235393959
	2	0.1729161412497188	0.03854192937514
	3	0.0958322824994376	0.01458385875028
4	0	0.6670691151510244	0.16646544242449
	1	0.3341382303020486	0.08293088484898
	2	0.1682764606040973	0.04086176969795
	3	0.0864580706248594	0.01927096468757
	4	0.0479161412497188	$7.2919293751 \cdot 10^{-3}$
5	0	0.6667672787877559	0.16661636060612
	1	0.3335345575755122	0.08323272121224
	2	0.1670691151510243	0.04146544242449
	3	0.0841382303020486	0.02043088484898
	4	0.0432290353124297	$9.6354823438 \cdot 10^{-3}$
	5	0.0239580706248594	$3.6459646878 \cdot 10^{-3}$
6	0	0.666691678710315	0.16665423113816
	1	0.3333836393938779	0.08330818030306
	2	0.1667672787877561	0.04161636060612
	3	0.0835345575755121	0.02073272121224
	4	0.0420691151510243	0.01021544242448
	5	0.0216147446992911	$4.8177411719 \cdot 10^{-3}$
	6	0.0119790353124297	$1.8229823438 \cdot 10^{-3}$

So we obtain formulas for evaluation atomic wavelets $wup_{m,k}(x)$ for any $m \in N$ and $k = 1,\dots,m$. We see that these functions are finite linear combinations of $g_{m,k}(x)$. It follows from (5) that we need only $g_{m,m}(x)$ to evaluate all other functions (see fig. 4–6). This property is very convenient for applications.

Thus atomic wavelets $wup_{m,k}(x)$ combine the following properties:

1) $wup_{m,k}(x)$ is infinitely differentiable (this implies form (3));

2) $\operatorname{supp} wup_{m,k}(x) = \left[0, \frac{3}{2^{k-1}}\right]$ (4);

3) for any natural m the system of functions

$$\left\{ mup_{2^m}(x-j);\ wup_{m,k}\left(x - \frac{j}{2^{k-1}}\right)\right\}_{k=1,\dots,m;\ j \in Z}$$

has 'good' approximation properties (see theorem 3).

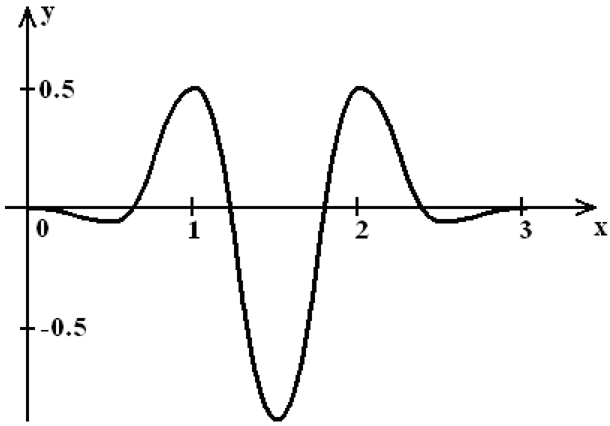


Fig. 4. Graph of $wup_{1,1}(x)$

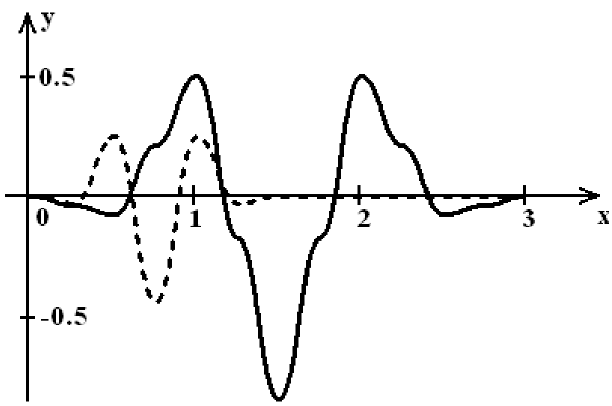


Fig. 5. Graphs of $wup_{2,1}(x)$ and $wup_{2,2}(x)$:
—— – $wup_{2,1}(x)$; – – – – – $wup_{2,2}(x)$

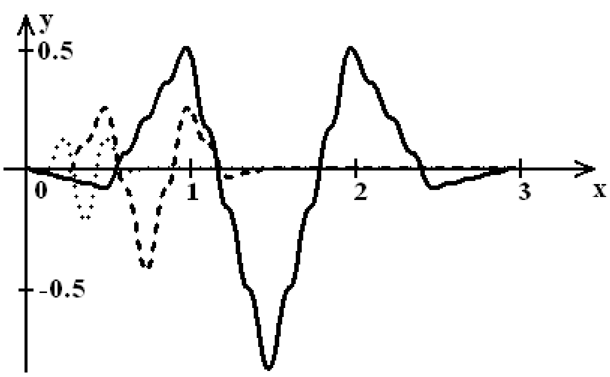


Fig. 6. Graphs of $wup_{3,1}(x)$, $wup_{3,2}(x)$ and $wup_{3,3}(x)$: —— – $wup_{3,1}(x)$;
– – – – – $wup_{3,2}(x)$; ······ – $wup_{3,3}(x)$

2.6. Examples

In this subsection we shall consider several examples. The function system

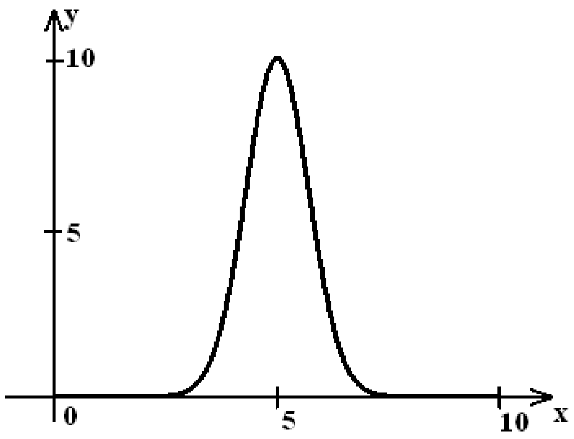
$$\widetilde{\Omega}_3 = \left\{ \mathrm{mup}_8(x-j), \mathrm{wup}_{3,k}\left(x - \frac{j}{2^{k-1}}\right) \right\}_{k=1,2,3;\ j \in Z}$$

will be used.

By $\tilde{f}(x)$ denote the best approximation of the function $f(x) \in L_2(R)$ by functions of the system $\widetilde{\Omega}_3$.

Example 1. Consider the function

$$f(x) = \begin{cases} 10 \cdot e^{-(x-5)^2}, & x \in [0,10] \\ 0, & x \notin [0,10] \end{cases}$$



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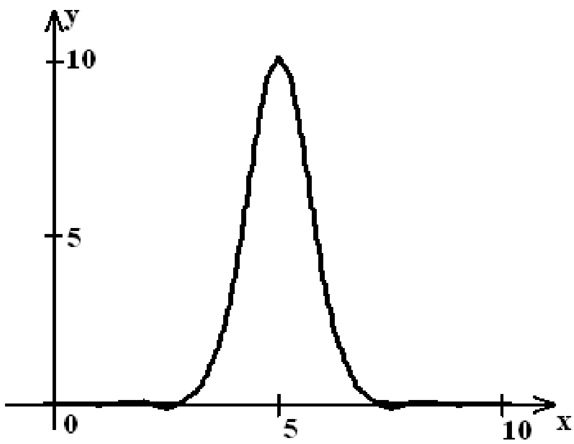
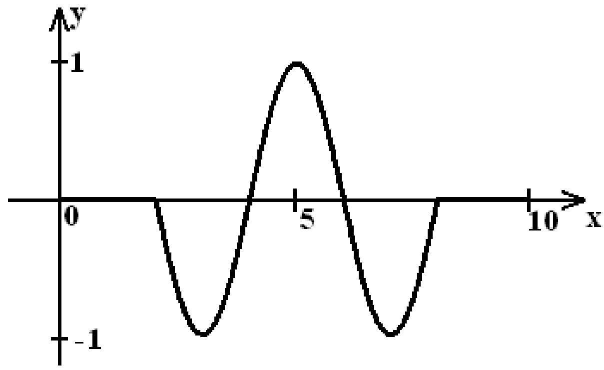


b

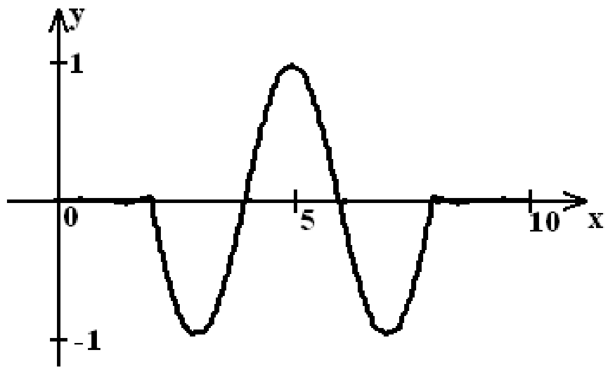
Fig. 7. Graphs of $f(x)$ and $\tilde{f}(x)$:
a – graph of f(x); b – graph of $\tilde{f}(x)$

Example 2. Consider the function

$$f(x) = \begin{cases} \sin\left(\frac{\pi}{2} \cdot x\right), & x \in [2,8] \\ 0, & x \notin [2,8] \end{cases}$$



a



b

Fig. 8. Graphs of $f(x)$ and $\tilde{f}(x)$:
a – graph of f(x); b – graph of $\tilde{f}(x)$

Example 3. Let $f(x) = \begin{cases} \frac{\sin(2\pi \cdot x)}{(x-5)^2+1}, & x \in [0,10] \\ 0, & x \notin [0,10] \end{cases}$

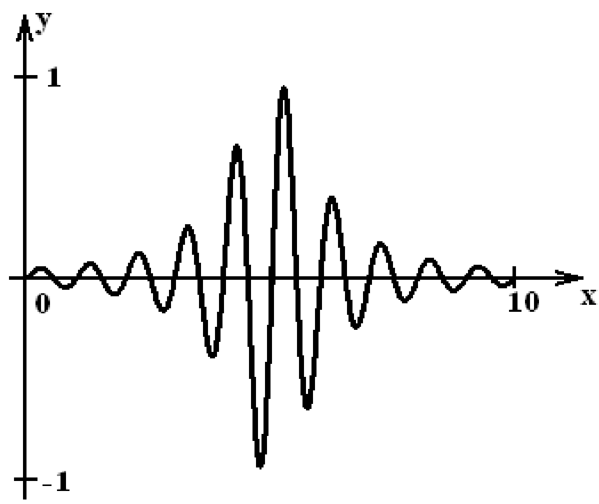


Fig. 9. Graph of the function $f(x)$

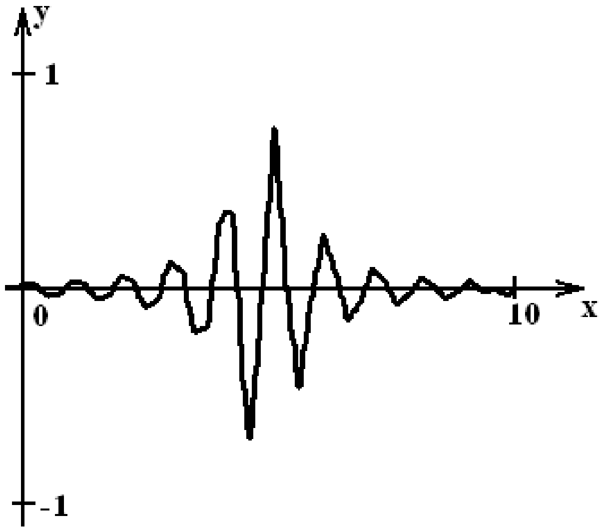


Fig. 10. Graph of the function $\tilde{f}(x)$

Example 4. Consider the function

$$f(x)=\begin{cases}\operatorname{sign}(x-5), & x\in[0,10]\\ 0, & x\notin[0,10]\end{cases}$$

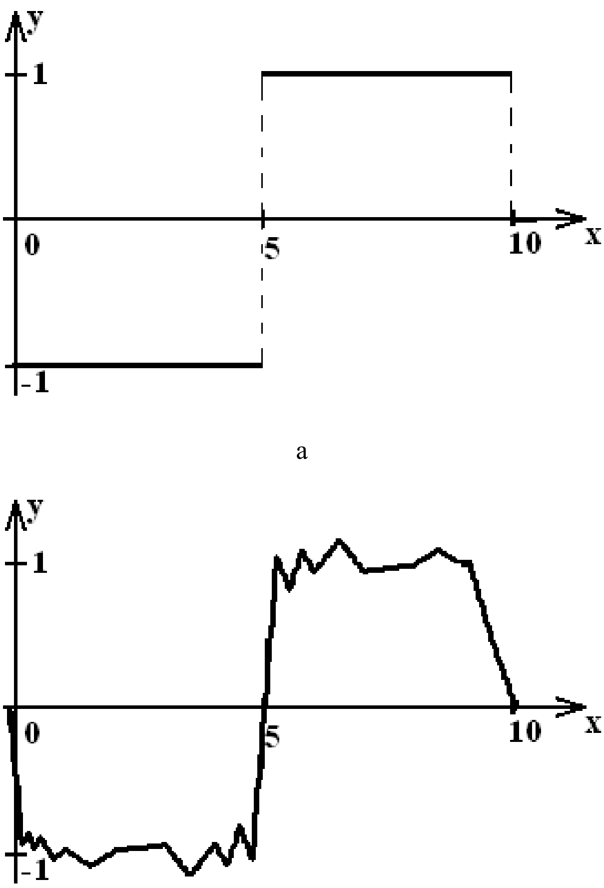


Fig. 11. Graphs of $f(x)$ and $\tilde{f}(x)$:
a – graph of f(x); b – graph of $\tilde{f}(x)$

2.7. Applications of the atomic functions

In this subsection we consider applications of atomic functions to the theory of generalized Taylor series.

By H_ρ denote the class of functions $f\in C^{\infty}_{[-1,1]}$ such that $\left|f^{(n)}(x)\right|\le c(f)\cdot\rho^n\cdot 2^{\frac{n(n+1)}{2}}$ for any $n=0,1,2,\dots$. It was proved in [25, 30] that if $f\in H_\rho$, $\rho\in[1,2)$, then $f(x)$ expands in the generalized Taylor series

$$f(x)=\sum_{n=0}^{\infty}\sum_{k\in N_n} f^{(n)}\left(x_{n,k}\right)\cdot\varphi_{n,k}(x),$$

where $N_0=\{-1,0,1\}$ and $x_{0,k}=k$, $k\in N_0$;

$$N_n=\left\{-2^{n-1},-2^{n-1}+1,\dots,2^{n-1}\right\},\ n\ne 0;$$

$$x_{n,k}=\frac{k}{2^{n-1}},\ n\ne 0,\ k\in N_n.$$

Basic functions $\varphi_{n,k}(x)$ are the finite linear combinations of translates of the atomic function $\mathrm{up}(x)$.

Let $H_{\alpha,s}$ ($s=2,3,4,\dots$) be the class of functions $f\in C^{\infty}_{[-1,1]}$ such that $\left|f^{(n)}(x)\right|\le c(f)\cdot\alpha^n\cdot 2^n\cdot(2s)^{\frac{n(n-1)}{2}}$ for any $n=0,1,2,\dots$. The generalized Taylor series for functions of these classes was introduced by V.A. Rvachev and G.A. Starets in [23]. It was shown in [23, 31] that if $f\in H_{\alpha,s}$ and $\alpha\in(1,2s)$, then $f(x)$ expands in the generalized Taylor series

$$f(x)=\sum_{n=0}^{\infty}\left(\sum_{k\in N_{s,n}} f^{(n)}\left(x_{s,n,k}\right)\cdot\varphi_{s,n,k}(x)+\right.$$

$$\left.+\sum_{p\in D_{s,n}}\Delta^2_{\frac{1}{s(2s)^n}}\left(f^{(n)};x^*_{s,n,p}\right)\cdot\psi_{s,n,p}(x)\right),$$

where $\Delta^2_h\left(f;x\right)=f(x+h)-2\cdot f(x)+f(x-h)$;

$$N_{s,0}=\{-1,0,1\}\ \text{and}\ x_{0,k}=k,\ k\in N_0;$$

$$N_{s,n}=\left\{-s\cdot(2s)^{n-1},\dots,s\cdot(2s)^{n-1}-1,s\cdot(2s)^{n-1}\right\};$$

$$x_{s,n,k}=\frac{k}{s\cdot(2s)^{n-1}}\ \text{for}\ k\in N_{s,n}\ \text{and}\ n\in N;$$

$$D_{s,n}=\left\{1,2,\dots,(2s)^{n+1}\right\}\setminus\{j\cdot s\},\ n=0,1,2,\dots;$$

$$x^*_{s,n,p}=-1+\frac{p}{s\cdot(2s)^n},\ n=0,1,2,\dots,\ p\in D_{s,n}.$$

The basic functions $\varphi_{s,n,k}(x)$ and $\psi_{s,n,p}(x)$ are defined by conditions: $\varphi_{s,n,k}\in H_{1,s}$, $\psi_{s,n,p}\in H_{1,s}$ and

$$\varphi^{(m)}_{s,n,k}\left(x_{s,m,j}\right)=\delta^m_n\cdot\delta^j_k,\ \psi^{(m)}_{s,n,p}\left(x_{s,m,j}\right)=0,$$

$$\Delta^2_{\frac{1}{s(2s)^m}}\left(\varphi^{(m)}_{s,n,k}; x^*_{s,m,q}\right) = 0,$$

$$\Delta^2_{\frac{1}{s(2s)^m}}\left(\psi^{(m)}_{s,n,p}; x^*_{s,m,q}\right) = \delta^m_n \cdot \delta^q_p,$$

where $n = 0,1,2,\ldots$, $m = 0,1,2,\ldots$, $k \in N_{s,n}$, $j \in N_{s,m}$, $p \in D_{s,n}$, $q \in D_{s,m}$ and δ^j_i is the Kronecker delta.

To evaluate $\varphi_{s,n,k}(x)$ and $\psi_{s,n,p}(x)$, we can use finite linear combinations of translates of the function $mup_s(x)$ [27]. It follows from (5) that $mup_{2^m}(x)$ is a finite linear combinations of translates of the function $g_{m,m}(x)$. This implies that $\varphi_{s,n,k}(x)$ and $\psi_{s,n,p}(x)$ can be evaluated using the values of $g_{m,m}(x)$.

Conclusions

Formulas for evaluation atomic wavelets $wup_{m,k}(x)$, which are infinitely differentiable functions with a local support, for the case $m \in N$ and $k = 1,\ldots,m$ have been obtained. These formulas can be used for the software development.

The atomic functions $Fmup_{m,n,k}(x)$ have been introduced. These functions possess nice properties and can be used in approximation of functions, numerical solution of differential and functional differential equations, optimal control problems and so on.

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Поступила в редакцию 17.01.2012

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АТОМАРНЫЕ ВЕЙВЛЕТЫ

И.В. Брысина, В.А. Макаричев

Рассмотрена задача существования и построения системы атомарных вейвлетов, состоящей из бесконечно дифференцируемых функций с компактным носителем. Получены формулы для вычисления атомарных вейвлетов. Приведены примеры использования атомарных вейвлетов для приближения некоторых функций. Рассмотрены финитные решения некоторых функционально-дифференциальных уравнений и их свойства. Введен новый класс атомарных функций. Приведены аппроксимационные свойства пространств конечных линейных комбинаций сдвигов атомарных функций.

Ключевые слова: вейвлет, функционально-дифференциальное уравнение, атомарная функция, атомарный вейвлет, финитная функция, бесконечно дифференцируемая функция, нестационарная система вейвлетов.

АТОМАРНІ ВЕЙВЛЕТИ

І.В. Брисіна, В.О. Макарічев

Розглянуто задачу існування та побудови системи атомарних вейвлетів, що складається з нескінченно диференційованих функцій з компактним носієм. Отримано формули для обчислення атомарних вейвлетів. Наведено приклади використання атомарних вейвлетів для наближення деяких функцій. Розглянуто фінітні розв'язки деяких функціонально диференціальних рівнянь та їх властивості. Введено новий клас атомарних функцій. Наведено апроксимаційні властивості просторів скінчених лінійних комбінацій зсувів атомарних функцій.

Ключові слова: вейвлет, функціонально-диференціальне рівняння, атомарна функція, атомарний вейвлет, фінітна функція, нескінченно диференційована функція, нестаціонарна система вейвлетів.

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