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BACKWARD INTERVAL MAPPING AS THE FOUNDATION OF DATA CODING ALGORITHM BASED ON CHAOTIC RECURRENT FUZZY MODELS

In this paper we investigate the dynamic systems, which are represented by recurrent Takagi-Sugeno rule bases that are widely used in many applications. The algorithm that shows the chaotic behaviour of Takagi-Sugeno models in the sence of Wigging-Devaney definition was developed. We use so-called "backward interval mapping" as the foundation of this algorithm. We describe an algorithm for data coding based on backward interval mapping method also.

recurrent fuzzy systems, chaos, transition function, backward mapping

1. Introduction

One of the main features in modelling dynamic systems is prediction. There are a lot of very simple dynamic systems which are unpredictable in principle and show chaotic behaviour [1, 2, 3]. There is an extensive literature for classical (unfuzzy) chaotic dynamic systems. We can represent a novel approach how to investigate fuzzy dynamic systems.

In the simplest case the recurrent Takagi-Sugeno (TS) fuzzy rule base of 0th order can be presented in the following form [2]:

$$R_1\text{: If } x_k = L_1 \text{ then } x_{k+1} = A_1,$$
$$R_2\text{: If } x_k = L_2 \text{ then } x_{k+1} = A_2 \quad (1)$$
$$\ldots$$
$$R_N\text{: If } x_k = L_N \text{ then } x_{k+1} = A_N,$$

where $x \in I = [0,1]$ is a scalar state variable, L_i are linguistic variables (terms), and the $A_i \in [0,1]$ are constants. The transition function $f(x): I \to I$ can be written in the form

$$f : x_k \to x_{k+1} \ . \quad (2)$$

It may be proved that three rules like (1) are necessary and sufficient for producing chaos mapping (if normality conditions are held for membership functions) [1].

Our aim is to find algorithm that shows chaotic behaviour of (2) in the sence of the definition Wigging and Devaney. As the next step of our investigations we research the applications of recurrent fuzzy models with chaotic behaviour.

Definition (Wigging and Devaney).

Mapping $f(x): I \to I$ is chaotic iif

1. it is topologically transitive, i.e., if there exists a $k > 0$ such that $f^k(U) \cap V \neq \varnothing$, where $f^k(U) = \{f^k(x) | x \in U\}$, for any pair of open sets, $U, V \subseteq I$.
2. it is sensitive to the initial conditions, i.e. if there exists a $\delta > 0$ such that $x \in I$ and any neighborhood N of x there exists a $y \in N$ and $n > 0$ such that $|f^n(x) - f^n(y)| > \delta$.
3. The periodic points of f are dense in I.

2. Backward Interval Mapping Method Application for Identification of Chaos in TS Models

Let us solve our task in reverse time. It allows to separate sets from step to step and investigate contracting mapping instead of expanding ones. Let $D_i = [a_i, a_{i+1}], i = \overline{1, N}$ are domains of x and determined from membership functions. It is clear that

$$\bigcup_{i=1}^{N} D_i = [a_1, a_N] = I \ , \ \bigcap_{i=1}^{N} D_i = \varnothing \ .$$

Then

$$\bigcup_{i=1}^{N} f_i\left(\{D_i\}\right) = I,$$
$$\bigcap_{i=1}^{N} f_i\left(\{D_i\}\right) \neq \varnothing. \quad (3)$$

Let us consider the inverse mapping $g(x) = f^{-1}(x)$. Let $f_i(x): D_i \to I$ is monotonic and continuous on each set D_i. For ($k-1$)th step we can write

$$x_{k-1} = f^{-1}(x_k). \quad (4)$$

In this case we will find the topological transitivity as

$$g^{-n}(U) \bigcap V \neq \varnothing. \quad (5)$$

Let us consider the three rule 0th order Takagi-Sugeno Model. Let us write sets as attribute mapping without figure brackets instead of scalar attribute x. Let

$$g(I) = \begin{cases} g_1(I) = I_1 \subset I, \\ g_2(I) = I_2 \subset I. \end{cases} \quad (6)$$

The conditions "$\subset$" are needed to do contracting mapping. It is clear that $I_1 = D_1, I_2 = D_2$, hence $I_1 \bigcup I_2 = I$ and $I_1 \bigcap I_2 = \varnothing$.

Let us consider next step of mapping g:

$$g^2(I) = \begin{cases} g_1(g(I)) = \begin{cases} g_1(g_1(I)) = I_{11} \subset I_1, \\ g_2(g_1(I)) = I_{21} \subset I_2, \end{cases} \\ g_1(g(I)) = \begin{cases} g_1(g_2(I)) = I_{12} \subseteq I_1, \\ g_2(g_2(I)) = I_{22} \subset I_2. \end{cases} \end{cases} \quad (7)$$

Let $I_{22} \neq \varnothing$ as well. It is easy to make sure that

$$I_{11} \bigcup I_{12} = I_1,$$
$$I_{21} \bigcup I_{22} = I_2. \quad (8)$$

Indeed, from (8) and monotonical and continuous mapping $g_i, i = 1,2$ we have

$$I_{11} \bigcup I_{12} = g_1(I_1) \bigcup g_1(I_2) = g_1(I_1 \bigcup I_2) = g_1(I) = I_1,$$
$$I_{21} \bigcup I_{22} = g_2(I_1) \bigcup g_2(I_2) = g_2(I_1 \bigcup I_2) = g_2(I) = I_2.$$

Equation (8) mean that if g is contracting mapping, i.e. $I_{11} \subset I_1$ then another new subsets like $I_{12} \subset I_1$ will be accorded to contracting mapping as well. The same claim is true for I_{21}, I_{22} sets too. It means as well that

$$I_{11} \bigcup I_{12} \bigcup I_{21} \bigcup I_{22} = I. \quad (9)$$

And the next Lemma gives us conditions for chaotic behaviour for the case $N = 2$.

Lemma

Let there be given $g(x) = (g_1(x), g_2(x))$ *where* $g_i, i = 1,2$ *are monotonous and continuous mapping on* $g(x): I \to I$ *and* $g(x)$ *is constructed in the next form*

$$g(I) = \begin{cases} g_1(I) = I_1 \subset I, \\ g_2(I) = I_2 \subset I, \end{cases} \quad (10)$$

where

$$I_1 \bigcup I_2 = I,$$
$$I_1 \bigcap I_2 = \varnothing$$

and

$$g^{K+1}(I) = \begin{cases} g_1\left(g^K(I)\right) = I_{1\{K\}}, \\ g_2\left(g^K(I)\right) = I_{2\{K\}}, \end{cases} \quad (11)$$

where $\{K\} = \{11...1, 11...2, ...22...2\}$ *is set of indexes length of K that were used for marking subset of* I *on* K^{th} *step and the next conditions are fulfilled*

$$g_1^{K+1}(I) \subset g_1^K(I), K = 0,1,...,$$
$$g_2^{K+1}(I) \subset g_2^K(I), K = 0,1,..., \quad (12)$$
$$g_2^K(I) \neq \varnothing, K = 0,1,...$$

then

$$I_{1\{K\}} \bigcup I_{2\{K\}} = I, \quad (13)$$

$$\left(\bigcap_{\{K\}} I_{1\{K\}}\right) \bigcap \left(\bigcap_{\{K\}} I_{2\{K\}}\right) = \varnothing \quad (14)$$

and $g(x) = (g_1(x), g_2(x))$ *is contracting mapping for the set* I *and all of it subsets.*

Proof

Mapping g is contracting mapping for the set I and all of it subsets that are generated. We will use method of induction. It is true that $I_1 \subset I, I_2 \subset I$. Then conditions (13),(14) are fulfilled.

$$I_1 \cup I_2 = I,$$
$$I_1 \cap I_2 = \varnothing.$$

Let conditions (12) - (14) is true for K^{th} step. According to (12)

$$I_{1,(11...1)_{k-1}} \subset I_{(11...1)_{k-1}},$$

$$I_{2,(22...2)_{k-1}} \subset I_{(22...2)_{k-1}}, I_{2,(22...2)_{k-1}} \neq \varnothing$$

and according to (13),(14)

$$I_{1\{K-1\}} \cup I_{2\{K-1\}} = I,$$

$$\left(\bigcap_{\{K-1\}} I_{1\{K-1\}}\right) \cap \left(\bigcap_{\{K-1\}} I_{2\{K-1\}}\right) = \varnothing.$$

Then for $K+1^{\text{th}}$ step

$$g^{K+1}(I) = \begin{cases} g_1\left(g^K(I)\right) = I_{1\{K\}}, \\ g_2\left(g^K(I)\right) = I_{2\{K\}}. \end{cases}$$

Set $I_{\{K+1\}}$ can be written in the form

$$I_{\{K+1\}} = \{I_{11\{K-1\}}, I_{21\{K-1\}}, I_{12\{K-1\}}, I_{22\{K-1\}}\},$$

where

$$I_{11\{K-1\}} = g_1\left(I_{1\{K-1\}}\right),\ I_{12\{K-1\}} = g_1\left(I_{2\{K-1\}}\right),$$

$$I_{21\{K-1\}} = g_2\left(I_{1\{K-1\}}\right),\ I_{22\{K-1\}} = g_2\left(I_{2\{K-1\}}\right).$$

Whence due to monotony and continuity of mapping on $g(x): I \to I$

$$I_{11\{K-1\}} \bigcup I_{12\{K-1\}} = g_1\left(I_{1\{K-1\}}\right) \bigcup g_1\left(I_{2\{K-1\}}\right) =$$

$$= g_1\left(I_{1\{K-1\}} \bigcup I_{2\{K-1\}}\right) = g_1(I) = I_1$$

and

$$I_{21\{K-1\}} \bigcup I_{22\{K-1\}} = g_2\left(I_{1\{K-1\}}\right) \bigcup g_1\left(I_{2\{K-1\}}\right) =$$

$$= g_2\left(I_{1\{K-1\}} \bigcup I_{2\{K-1\}}\right) = g_2(I) = I_2 \ .$$

Therefore (13) is true

$$I_{1\{K\}} \bigcup I_{2\{K\}} = I \ .$$

Let us prove (14). It is clear that

$$\left(\bigcap_{\{K\}} I_{1\{K\}}\right) = \left(\bigcap_{\{K-1\}} g_1\left(I_{1\{K-1\}}\right)\right) \bigcap \left(\bigcap_{\{K-1\}} g_1\left(I_{2\{K-1\}}\right)\right) =$$

$$= g_1\left(\left(\bigcap_{\{K-1\}} I_{1\{K-1\}}\right) \bigcap \left(\bigcap_{\{K-1\}} I_{2\{K-1\}}\right)\right) = \varnothing,$$

$$\left(\bigcap_{\{K\}} I_{2\{K\}}\right) = \left(\bigcap_{\{K-1\}} g_2\left(I_{1\{K-1\}}\right)\right) \bigcap \left(\bigcap_{\{K-1\}} g_2\left(I_{2\{K-1\}}\right)\right) =$$

$$= g_2\left(\left(\bigcap_{\{K-1\}} I_{1\{K-1\}}\right) \bigcap \left(\bigcap_{\{K-1\}} I_{2\{K-1\}}\right)\right) = \varnothing.$$

Hence

$$\left(\bigcap_{\{K\}} I_{1\{K\}}\right) \cap \left(\bigcap_{\{K\}} I_{2\{K\}}\right) = \varnothing \ .$$

According to (13)

$$I_{1,(11...1)_k} \subset I_{(11...1)_k} \ ,\ I_{2,(22...2)_k} \subset I_{(22...2)_k}, \qquad (15)$$

hence

$$diam\left(I_{1,(11...1)_k}\right) < diam\left(I_{(11...1)_k}\right),$$
$$diam\left(I_{2,(11...1)_k}\right) < diam\left(I_{(22...2)_k}\right).$$

The rest subsets of $I_{\{K+1\}}$ have less diameters too.

Because of (15) the rest of $I_{\{K+1\}}$ will be no more then $I_{\{K\}}$, so $I_{\{K+1\}} \subseteq I_{\{K\}}$. Weak inclusion can be explained that some of subsets during the mapping may be empty.

Therefore we can symbolically write

$$\lim_{n\to\infty}\left(diam\left(I_{\{K\}}\right)\right) = 0 \ .$$

Property (14) defines that the mapping $g(x) = (g_1(x), g_2(x))$ on the subsets of I is also dense on I .

We can interchange numbering in the mapping $g(x) = (g_1(x), g_2(x))$, sets I_1, I_2 and appropriate conditions (8) - (14).

Theorem

If $g(x) = (g_1(x), g_2(x))$ *is fulfilled to* ***Lemma*** *conditions then* $f(x) = g^{-1}(x)$ *is chaotic in sense of Wigging.*

Proof

Let $I_1 = g_1(I) = D_1, I_2 = g_2(I) = D_2$ where D_1, D_2 are domains for monotonous and continuous mapping $f(x) = g^{-1}(x)$. Then

$$I_1 \bigcup I_2 = I,$$
$$I_1 \bigcap I_2 = \varnothing.$$

Because mapping $g(x) = (g_1(x), g_2(x))$ is dense and we can find appropriate sets V and U on arbitrary level K . Namely, we will find such level, that some inequalities are fulfilled

$$\bigcup_{i=i_1}^{i_m-1} I_i^K \subset V \subset \bigcup_{i=i_1}^{i_m} I_i^K \ ,\ \bigcup_{i=i_2}^{i_n} I_i^K \subset U \ .$$

For each subset on level K there exists parent subset on level $K-1$ and so on to 1[st] set I. The closest level Λ to K that is produced by mapping of some $I_j^K \subset U$ and $\bigcup_{i=i_1}^{i_m} I_i^K \subset f^{\Lambda}\left(I_j^K\right)$ will be satisfied is the level that is defined transitive dependence.

In worse case $\Lambda = K$ i.e. $f^{\Lambda}\left(I_j^K\right) = I$.

Second condition of Wigging's properties of chaos is sensitivity to initial conditions. Mapping $f(x) = g^{-1}(x)$ is sensitive to the initial conditions, i.e. if there exists a $\delta > 0$ such that $x \in I$ and any neighborhood N of x there exists a $y \in N$ and $n > 0$ such that $\left|f^n(x) - f^n(y)\right| > \delta$.

One can find such level K and some little intervals I_x^K and I_y^K that $f^n\left(I_x^K\right) = I_x^{K-n}$ $f^n\left(I_y^K\right) = I_y^{K-n}$ and $I_x^{K-n} \bigcap I_y^{K-n} = \varnothing$ according to (13) and the condition $\left|f^n(x) - f^n(y)\right| > \delta$ will be fulfilled.

3. Example of Coding and Reconstruction of Chaotic Bit Series using Backward Interval Mapping

When we have triangular membership functions the mapping (2) is isomorphic to well known tent mapping [3, 4]. In such case we can rewrite the mapping (2) as the slopping tent mapping [5] $f(x): I \to I$, $I = [0,1]$

$$x_{k+1} = \begin{cases} f_1(x_k) = \dfrac{1}{\lambda} x_k, & if \quad 0 \le x_k \le \lambda, \\ f_2(x_k) = \dfrac{1}{\lambda - 1} x_k + \dfrac{1}{1-\lambda}, & if \quad \lambda \le x_k \le 1, \end{cases} \tag{16}$$

where $x_k \in [0,1]$, $\lambda \in (0,1)$ (fig. 1).

Let us consider the following bit sequence

$$C = \{c_i\}_{i=1}^N \tag{17}$$

with length N, $c_i \in \{0,1\}$.

It is necessary to restore the source sequence (17) as a bit sequence

$$\tilde{C} = \{\tilde{c}_i\}_{i=M}^N \tag{18}$$

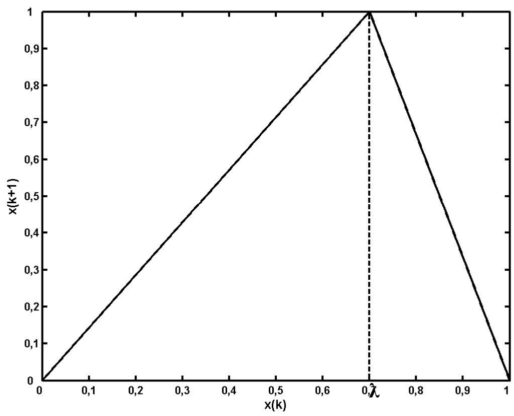

Fig. 1. Slopping mapping

according to the rule

$$\tilde{c}_i = \begin{cases} 1, & if \quad x_i \ge \lambda, \\ 0, & if \quad x_i < \lambda \end{cases} \tag{19}$$

for $i = \overline{M,N}$, $M \le N$.

Namely, it is necessary to find such value x_M that gives the same values for restoring sequence like in the source one. Besides, it is important to find such value of λ, that gives maximum members of restored sequence. The best case is when $M = 1$.

Underlying methodology is based on backward interval mapping.

The mapping $g(x) = f^{-1}(x)$ is contracting mapping if $f(x)$ is chaotic.

For $(k-1)$[th] step we can write

$$x_{k-1} = g(x_k). \tag{20}$$

For slopping tent mapping (16) we have next backward mapping [5]

$$x_k = \begin{cases} g_1(x_{k+1}) = \lambda x_{k+1}, \\ g_2(x_{k+1}) = (\lambda - 1)x_{k+1} + 1. \end{cases} \tag{21}$$

Let us consider the action of this mapping when argument of function g is interval (fig. 2). For initial interval I we have

$$g(I) = \begin{cases} g_1(I) = I_1 = [0, \lambda] \subset I, \\ g_2(I) = I_2 = [\lambda, 1] \subset I. \end{cases} \tag{22}$$

The following iterative procedure for backward interval mapping takes place. Let choose initial interval according to rule

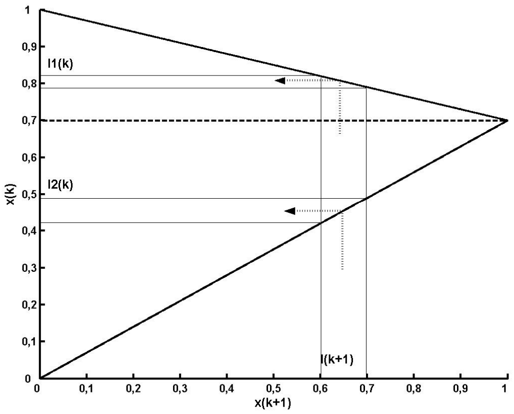

Fig. 2. Backward interval slopping mapping

$$I_N = \begin{cases} [\lambda, 1], & if \;\; c_N = 1, \\ [0, \lambda], & if \;\; c_N = 0. \end{cases} \quad (23)$$

Then define the possible transitions with the backward interval mapping

$$\widetilde{I}_{N-1} = g(I_N) = \begin{cases} g_1(I_N) \subset [0, \lambda], \\ g_2(I_N) \subset [\lambda, 1]. \end{cases} \quad (24)$$

The next interval is made more precise in according to the value c_{N-1}:

$$I_{N-1} = \begin{cases} g_2(I_N), & if \;\; c_{N-1} = 1, \\ g_1(I_N), & if \;\; c_{N-1} = 0. \end{cases} \quad (25)$$

The procedure (24), (25) is repeated until obtaining the limit of accuracy

$$diam(I_{M-1}) = \varepsilon \,. \quad (26)$$

Then any values $x_M \in I_M$ restore the source sequence (17) with forward mapping (16) form M number of sequence.

Parameter ε is machine accuracy. In general case it is $\varepsilon = 10^{-15}$ (in MatLab, Delphi programming environments, so on).

So we can not restore arbitrary sequence with backward methods (only for moment when interval became ε - length).

Hence, we have a $\{0,1\}$ - sequence and a "machine" which is able to reproduce this sequence starting with certain real x0. Thus we have to store only x0 (and the "machine", i.e. the tent mapping). If we use slopping tent map we need to extra store λ as well and number of elements in series. If we use simple tent mapping it is necessary to store only x0. The initial sequence of $\{0,1\}$ is restored according algorithm - if chaos (0,1) sequence is greater then λ we write 'one' else 'zero'. In simple tent map lambda is 0.5.

Conclusion

Obviously that critical factor of the algorithm is accuracy of software and hardware. It depends on sensitivity to initial conditions of chaotic mapping. The novelty of approach proposed consists in substitution of chaotic bit series by initial value x_M that can reconstruct original orbit with mapping (16). In future we plan to expand the proposed approach to the TS model of 1st order. Approach proposed can be used for coding and data ciphering as well.

References

1. Li T.Y., Yorke J.A. Period three implies chaos // Amer. Math. Monthly. – 1975. – Vol. 82. – P. 985 – 992.
2. Kempf R., Adamy J. Regularity and chaos in recurrent fuzzy systems // Fuzzy Sets and Systems. – 2003. – Vol. 140. – P. 259 – 284.
3. Kloeden P.E. Cycles and chaos in higher dimensional difference equations // Proceedings of the 9th Internat. Conf. Nonlinear Oscillations. – K.: Naukova Dumka. – 1984. – Vol. 2. – P. 184 – 187.
4. Kloeden P.E. Chaotic iterations of fuzzy sets // Fuzzy Sets and Systems. – 1991. – Vol. 42. – P. 37 – 42.
5. Sokolov A., Wagenknecht M. Investigation of chaotic behavior of fuzzy Takagi-Sugeno models with regard to simulation and control of technological processes // Scientific Report, Univ. of Zittau/Goerlitz, IPM. – 2003.

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