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Pavlo LUKIANOV, Lin SONG

National Aviation University, Kyiv, Ukraine

FLOW DEVELOPMENT REGION IN THE BOUNDARY LAYER: TWO-COMPONENT MOLECULAR VISCOSITY AND PARTIAL SLIP

The subject *of this study is the flow development region of laminar incompressible fluid flow in the boundary layer. This flow is an example where a direct application of the Navier-Stokes equations of gradient-free laminar incompressible fluid flow, in which the molecular viscosity is assumed to be a constant value independent of spatial coordinates, leads to a redefinition of the mathematical model. It is about the fluid boundary layer in the region of flow establishment in the motion problem of a semi-infinite plane, where the pressure gradient is zero. There is a situation when the number of equations is equal to three (two equations of momentum conservation and the equation of continuity), and the number of unknowns is equal to two - the number of the speed component. As a logical solution to the obtained inconsistency, it is proposed, as was already done for the problem of stationary motion of a plane and the problem of acceleration of a plane, to depart from the false statement about the constancy of molecular viscosity in the gradient-free boundary layer of an incompressible flow and consider molecular viscosity as a function of spatial coordinates. The need to consider the variable nature of molecular viscosity led to the discovery of another flaw in the Navier-Stokes theory. This non-trivial flaw was discovered during the application of the original numerical analytical method for solving the flow development region problem. The Navier-Stokes equations are supplemented by boundary conditions. The most important condition is the condition of fluid non-slipping on the surface of a solid body, which, by the way, does not follow any physical law. As a result, on the surface of a half-plane (or a moving body), the component of the velocity, which coincides with the direction of motion, has a constant value equal to the velocity of the body. It immediately follows from the continuity equation that the normal derivative of the normal component of the velocity must be equal to zero along the surface of the plane (body), since the longitudinal derivative of the velocity becomes zero. However, it is quite obvious that the velocity component normal to the surface of the plane (body) changes across the boundary layer in the region of current development, which indicates the presence of a normal gradient (both components) of the velocity. The conflict or contradiction is overcome by moving away from the generally accepted condition of non-slipping to the condition of partial non-slipping, or essentially the presence of sliding. Even with the sudden braking of any vehicle, the complete stop does not occur instantly, but after some finite time and distance, so in the case of the motion of a body in a stationary fluid, there is not an instant sticking, but a gradual one - from complete sliding, when a particle of liquid has just met a moving plane (body), to complete non-slipping at the end (and further) of the flow development region.* ***Research methods****. This work uses purely theoretical methods based on the use of calculus of variations, laws of physics, and ideas from everyday life.* ***Conclusions.*** *An improved model of a viscous Newtonian fluid in the area of flow development in the boundary layer was derived. On the basis of assumptions about the variable nature of the molecular viscosity, which already has two components, and the departure from the non-slipping condition, analytical solutions for both components of the velocity and both components of the molecular viscosity were obtained. A comparison of the obtained results with the results of other studies is presented.*

Keywords: *airplane; helicopter; area of flow development; boundary layer; partial slip.*

Introduction

As a body moves in a fluid, contact friction occurs between the fluid and the surface of the body. As a result of this friction, the fluid is not able to instantly accelerate to finite velocity. This refers to the flow around the front part of the aircraft fuselage, its wings, helicopter blades, and other parts of bodies that move in a still fluid. Therefore, the velocity field is generated only in a thin boundary layer, the correct calculation of which is extremely important for calculating the friction force [1]. Based on knowledge about the structure of the boundary layer on the wing, the optimal wing profile is determined [2].

More than 200 years ago, in 1822, Navier derived the equations that describe the deformation of a solid body and the motion of an incompressible fluid. But these equations received general recognition after the publication of Stokes' theory a little later, in 1845 [3]. Stokes was also based on the mathematical analogy between the equations describing the motion of a viscous fluid and the processes of heat transfer - the Fourier theory. However, Stokes did not take into account one important assumption of Fourier's theory: Fourier's second law, where the coefficient of thermal conductivity is considered a constant value, has no place (is not valid) in the region near the interface of different media (for details, see [4]). And this is because it is in the boundary layer that the behavior of one phase is interconnected with the behavior of another. In the case of the flow under consideration, the behavior of the fluid in the boundary layer is determined by the presence of a solid surface of the moving body. And this, in turn, does not correspond to homogeneity and spatial anisotropy, which guarantee, according to Fourier, the constancy of the thermal conductivity coefficient. Therefore, by means of simple logic, an important conclusion was obtained: for incompressible laminar fluid flow in the boundary layer, it is necessary to depart from the concept of constancy of molecular viscosity and keep in mind that viscosity, generally speaking, can be a function of spatial coordinates.

Now let's return to our problem. If you try to shed light and understand what is already known about the flow development region, the first surprise will be very funny: even now, this problem is not solved directly, but as an asymptotic transition of a non-stationary flow to a stationary one at very large time values. This fact intrigued us, and we made an attempt to directly solve the specified problem using known analytical methods. Moreover, the presence of an inverse relationship in the system fluid – moving body gives every reason to use the appropriate mathematical apparatus, which is the calculus of variations.

The joint application of the calculus of variation method and information about the specifics of the movement of an incompressible fluid in the boundary layer led to a clear understanding the fact that in the flow development region the boundary condition of non-slipping does not take place. This fact prompted us to study the available information in modern sources about the fluid slipping past the surface of a solid body. Let's briefly dwell on the sources on which our attention was focused.

First of all, it should be noted that Navier himself raised the issue of fluid sliding on the surface of a solid body a year after the publication of his equations [5]. Therefore, Navier did not reject the possibility of non-meeting of the non-slipping condition, which Stokes actually postulated -- without any physical grounds.

In the article [6] it is noted that nano -bubbles (air) were experimentally observed on smooth water-repellent surfaces. In addition, cracks (of small scale) can serve as places for the accumulation of bubbles in the case of the use of fluids that partially wet the surface. These bubbles can provide a zero shear stress boundary condition and significantly reduce the friction generated by the solid boundary.

Also worthy of attention is the work [7], which considers the boundary conditions of (effective) slipping by the method of simulation (modeling) of molecular dynamics. An interesting fact is that the local boundary conditions, both on wetted and non-wetted regions, are characterized by finite Navier slip scale lengths. But the main thing for our work is the presence of liquid along the solid surface. The study of partial slipping is devoted to work [8]. This work, according to the authors themselves, was motivated by the violation of the condition of fluid flow non-slipping in the millimeter scale domain. It seemed that the viscosity and the non-slipping condition should play a significant role in the balance of forces. But it is not quite so. Among the important conclusions of the cited work, it should be noted the dependence of diffusion on the local conditions of the wall, which correlates with the results [4,9] about the spatial dependence of molecular diffusion.

The presence of fluid slipping past a solid surface is studied in [10] from the point of view of stream function solutions for some boundary conditions of the contact line. In other words, the interaction of a fluid with a solid body is identified with some contact line. But the boundary conditions considered in [11] are as follows: Navier slip, super-slip, and the generalized Navier boundary condition. So, without going into the details of the cited work, we can, however, confidently assert the presence of slippage.

In an effort to reduce friction, scientists and engineer resort to various means. One of them is the use of folds (corrugation) (grooves) and accumulation, as already clear from the previously cited works, of bubbles or fluid there, which leads to partial, or as it is also called, effective slipping. Research in this area is presented in [12].

Let's finish the short review of works with the article [13]. This work is devoted to the movement of the contact line between two immiscible fluids. But the main thing that it states is that the Stokes equations do not allow to describe the specified flow when two different velocity values occur on two different boundaries of the contact line.

So, the given brief overview of modern works confidently asserts about the possible violation of the condition of non-slip on a solid body of a viscous fluid flow. We will only confirm this and show further that there are also flows within the Newtonian fluid model, where the condition of complete non-slip simply cannot be met and, thus, must also be replaced by the condition of partial slipping.

Problem formulation

Consider a semi-infinite plane $x \in (-\infty;0]$ moving with some constant velocity in the positive direction of

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the abscissa axis (see Fig. 1). At each subsequent moment, this plane involves fluid particles that were at rest before that into motion. *The task is to determine, using analytical methods, the components of velocity and molecular viscosity in the region of development of incompressible laminar fluid flow in the boundary layer during the motion of a semi-infinite plane in a stationary fluid.*

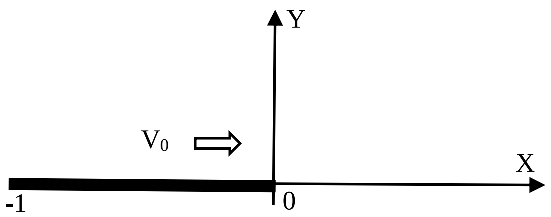

Fig. 1. Motion of a semi-infinite plane with a constant velocity in a fluid at rest at infinity

1. Expansion of the existing model of viscous laminar fluid flow development region in boundary layer

According to the existing theory, such motion should be described by the Navier-Stokes equations. For our problem we obtain

$$\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}=\mu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}\right); \quad (1)$$

$$\frac{\partial V_y}{\partial t}+V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}=\mu\left(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}\right); \quad (2)$$

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}=0. \quad (3)$$

For incompressible laminar flow, the molecular viscosity is a constant and thus we have three equations for the two unknown functions. That is, from a mathematical point of view, the problem is overdetermined. What was not taken into account? The only possible way to overcome this discrepancy is to depart from the concept of constant viscosity in laminar incompressible fluid flow, which simply means:

$$\mu \neq \mathrm{Const}. \quad (4)$$

Taking into account relation (4), equations (1), (2) take the form:

$$\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}=$$
$$=\frac{\partial \mu}{\partial x}\frac{\partial V_x}{\partial x}+\frac{\partial \mu}{\partial y}\frac{\partial V_x}{\partial y}+\mu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}\right); \quad (5)$$

$$\frac{\partial V_y}{\partial t}+V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}=$$
$$=\frac{\partial \mu}{\partial x}\frac{\partial V_y}{\partial x}+\frac{\partial \mu}{\partial y}\frac{\partial V_y}{\partial y}+\mu\left(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}\right). \quad (6)$$

The system of equations (3), (5), (6) is now entirely defined: three different equations correspond to three unknown functions. It seemed that all troubles were over. But no - the main intrigue of this work is still ahead. To solve the problem, differential equations must be supplemented with initial and boundary conditions. Since the changing nature of the molecular viscosity inside the laminar boundary layer of an incompressible flow was first described on the basis of a stationary flow [4], then we will limit ourselves to the stationary flow and instead of (5), (6) we will consider

$$V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}=\frac{\partial \mu}{\partial x}\frac{\partial V_x}{\partial x}+\frac{\partial \mu}{\partial y}\frac{\partial V_x}{\partial y}+$$
$$+\mu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}\right); \quad (7)$$

$$V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}=\frac{\partial \mu}{\partial x}\frac{\partial V_y}{\partial x}+\frac{\partial \mu}{\partial y}\frac{\partial V_y}{\partial y}+$$
$$+\mu\left(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}\right). \quad (8)$$

The system of equations (3), (7), (8) is written in dimensional form. Without knowing in advance the spatial extent of the flow establishment area, it is not convenient to formulate the boundary conditions. Therefore, assuming the anisotropy of spatial scales

$$\delta=\frac{l_y}{l_x}<<1,$$

consider the dimensionless analog of equations (3), (7), (8). This analogue is such a system

$$V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}=\delta^2\left(\frac{\partial \mu}{\partial x}\frac{\partial V_x}{\partial x}+\mu\frac{\partial^2 V_x}{\partial x^2}\right)+$$

$$+\frac{\partial\mu}{\partial y}\frac{\partial V_x}{\partial y}+\mu\frac{\partial^2 V_x}{\partial y^2}; \qquad (9)$$

$$V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}=\delta^2\left(\frac{\partial\mu}{\partial x}\frac{\partial V_y}{\partial x}+\mu\frac{\partial^2 V_y}{\partial x^2}\right)+$$

$$+\frac{\partial\mu}{\partial y}\frac{\partial V_y}{\partial y}+\mu\frac{\partial^2 V_y}{\partial y^2}; \qquad (10)$$

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}=0. \qquad (11)$$

Now, after the transition to dimensionless quantities, the flow development region has the form of a square

$$[x]\times[y]=[0;1]\times[0;1].$$

Hereafter, we will neglect terms of the second order of smallness - those containing δ^2. In this case, (9), (10) take on a simpler form:

$$V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}=\frac{\partial\mu}{\partial y}\frac{\partial V_x}{\partial y}+\mu\frac{\partial^2 V_x}{\partial y^2}; \qquad (12)$$

$$V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}=\frac{\partial\mu}{\partial y}\frac{\partial V_y}{\partial y}+\mu\frac{\partial^2 V_y}{\partial y^2}. \qquad (13)$$

The main boundary condition of the viscous flow model is non-slip on the surface of a solid body. This condition postulates that the velocity of fluid particles touching the surface of a solid must be equal to the velocity of the body. In other words, if the body does not move, then the non-slip condition means zero fluid velocity on the surface of the body; if the body moves in a still fluid, then the velocity of the fluid on the surface of the solid body must be equal to the velocity of the body. We will show that this non-slip condition is not met in the flow development region. Indeed, from the continuity equation (3 or 11) and the non-slip condition, it follows that along the entire flow development region, we have

$$\frac{\partial V_x}{\partial x}=0\Rightarrow\frac{\partial V_y}{\partial y}=0. \qquad (14)$$

It immediately follows from relation (14) that the normal component of the velocity in the region of the boundary layer does not change across this layer. It is known that this is not so. So, we have reached another contradiction of Stokes theory – *the impossibility of fully complying with the condition of no slipping on the surface of a solid body in the region of flow development.*

How to understand the resulting contradiction? For this, let's turn to physics as a science and life experience. We know very well that Newton's second law does not allow a body to instantly acquire a finite velocity - this requires an infinitely large force. Does the force of friction of a liquid against a solid belong to the category of infinitely large? Obviously not. We know from life experience that braking any vehicle requires a finite time, which corresponds to a finite path - to a complete stop of motion. By analogy, during the friction of a fluid against a body, an instantaneous transition from the rest of the fluid to a finite (and not small in the case of aviation and space technic) velocity value cannot, in principle, occur. The only possible is a gradual increase in velocity: during friction against a surface, the fluid slides past this surface, gradually gaining speed and the acceleration process ends at the end of the development region. Therefore, instead of the non-slip condition, we must satisfy the fluid acceleration condition - an increase in its velocity on the surface of the solid body from zero (at the beginning of the flow development region) to the maximum body velocity (at the end of the development region). How to do it, i.e. how, according to which formula to set the law of growth of velocity on the surface of the body - is not yet known. Let's try to solve this problem. As for other boundary conditions in the region of flow establishment, they are obvious: at the outer boundary of the boundary layer, the velocity asymptotic tends to zero, and after passing the development region – the asymptotic decaying of the normal component of the velocity everywhere across the boundary layer. And also the equality to zero of the longitudinal derivatives for of all magnitudes: velocity and viscosity components.

As mentioned above, attempts to solve the problem by numerical and analytical methods led to the realization of the discussed inconsistency of the non-slip boundary condition. However, solving the problem turned out to be possible with the help of analytical methods.

2. Application of calculus of variations for the analytical solution of the problem of establishing the flow during steady motion of a semi-infinite plane in a fluid at rest

As in works [4, 9], we will assume that the fluid flow rate caused by friction against the surface of the moving body is extreme. Most likely, minimal: during

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interaction with a solid body, the fluid tries to be involved in the movement as little as possible. This is "as little as possible" and there is a minimal flow rate. Fluid consumption is determined by the longitudinal component of the velocity and is described by the functional:

$$J=\int_0^1 V_X\left(\frac{\partial V_X}{\partial x},\frac{\partial V_X}{\partial y},\frac{\partial V_y}{\partial x},\frac{\partial V_y}{\partial y}\right)dy. \qquad (15)$$

It is no coincidence that the integral expression (15) is a function of the gradients of the velocity components. The fact is that the motion of the fluid in the problem under consideration is completely dependent on the tangential stresses, which are known to be determined by the gradients of the velocity field. After applying the procedure described in detail in [4], we obtain the Euler equation of the calculus of variations for the longitudinal component of the velocity for the necessary condition of the extreme of the functional (15). This equation has the following form:

$$-\frac{\partial}{\partial x}\left(\frac{\partial V_X}{\dfrac{\partial V_X}{\partial x}}\right)-\frac{\partial}{\partial y}\left(\frac{\partial V_X}{\dfrac{\partial V_X}{\partial y}}\right)=0. \qquad (16)$$

We will search the solution of equation (16) using the method outlined in [9]. The essence of this method is that the asymptotic tendency of the flow, after passing through the development region, to the form corresponding to the motion of an infinite plane, allows (16) to be split into two equations:

$$\frac{\partial}{\partial x}\left(\frac{\partial V_X}{\dfrac{\partial V_X}{\partial x}}\right)=\frac{\partial}{\partial y}\left(\frac{\partial V_X}{\dfrac{\partial V_X}{\partial y}}\right)=0. \qquad (17)$$

Using the invariance of the first differential, we transform (17) into the form

$$\frac{\partial}{\partial x}\left(\frac{\partial V_X}{\partial x}\frac{1}{\dfrac{\partial^2 V_X}{\partial x^2}}\right)=\frac{\partial}{\partial y}\left(\frac{\partial V_X}{\partial y}\frac{1}{\dfrac{\partial^2 V_X}{\partial y^2}}\right)=0. \qquad (18)$$

The method of distribution of variables [9] can be applied to the system of differential equations (18). According to this method,

$$V_X=X_X(x)Y_X(y). \qquad (19)$$

Substituting (19) into (18), we get:

$$\frac{\partial}{\partial x}\left(\frac{\partial X_X}{\partial x}\frac{1}{\dfrac{\partial^2 X_X}{\partial x^2}}\right)=\frac{\partial}{\partial y}\left(\frac{\partial Y_X}{\partial y}\frac{1}{\dfrac{\partial^2 Y_X}{\partial y^2}}\right)=0. \qquad (20)$$

The general solution of system (20) has the following form

$$X_X(x)=A_X+B_X\exp\left(\frac{x}{C_{1x}}\right),$$
$$Y_X(x)=A_y+B_y\exp\left(\frac{y}{C_{1y}}\right). \qquad (21)$$

According to the known information on the structure of the velocity field in the case of an infinite plane [4], as well as the need for the flow to reach the asymptotic regime without slipping, from (21) we have:

$$V_x(x,y)=(1-\exp(\alpha x))\exp(-\alpha y), \quad \text{where } \alpha\approx 5. \qquad (22)$$

For the second, normal component of the velocity field, the necessary condition for the extremum of the functional must also be fulfilled (Fig. 2). She looks like this

$$\frac{\partial}{\partial x}\left(\frac{\partial V_X}{\partial x}\frac{1}{\dfrac{\partial^2 V_y}{\partial x^2}}\right)=\frac{\partial}{\partial y}\left(\frac{\partial V_X}{\partial y}\frac{1}{\dfrac{\partial^2 V_y}{\partial y^2}}\right)=0. \qquad (23)$$

The system of equations (23) can, theoretically, be solved analytically. But, as it turns out, it is easier to find $V_y(x,y)$ from the continuity equation and show that the solution obtained in this way satisfies (23). Let's do it. After substituting (22) into (11), we obtain

$$\frac{\partial V_y}{\partial y}=-\frac{\partial V_x}{\partial x}=\alpha\exp(\alpha(x-y)). \qquad (24)$$

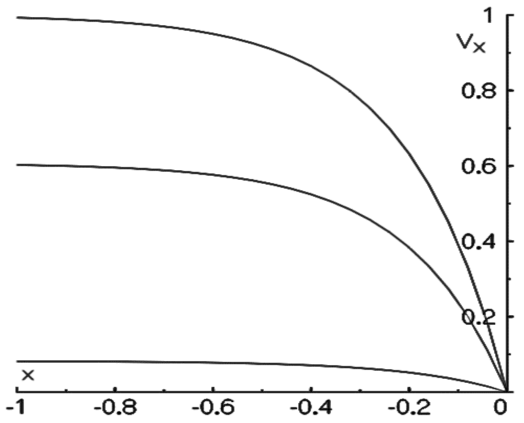
The solution of (24) is

$$V_y=\exp(\alpha(x\text{-}y))+F_1(x). \qquad (25)$$

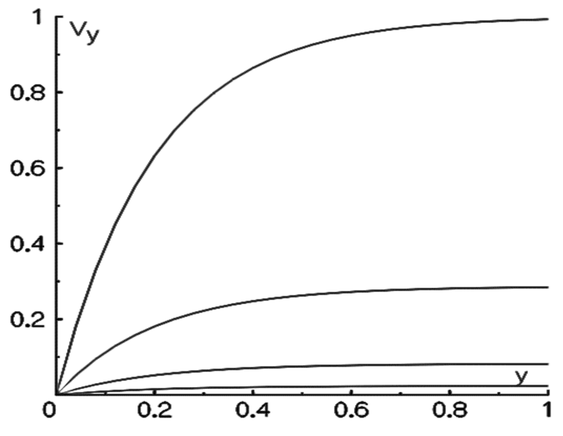
Taking into account the equality of the zero normal component of the velocity, it is finally obtained

$$V_y=\exp(\alpha x)(1-\exp(-\alpha y)). \qquad (26)$$

It should be pointed out that the solution (26) does not tend to zero, for each fixed value of x in flow development region, as is the case with the longitudinal component of the velocity. And this does not contradict the fact that at the boundary of the boundary layer, the velocity practically decreases to zero: let's remember that the normal component of the velocity is many (hundreds or even more) times smaller than the longitudinal component. Therefore, any velocity measurements will actually correspond to the decreasing of the longitudinal component. A similar result was obtained by Blasius [14] for the inverse problem. There, too, the normal component of the velocity becomes constant as y increases. It should be especially noted that expressions (22) and (26) satisfy the system of equations (23), which confirms their validity: not only the continuity equation, but also both necessary conditions for the extreme of the fluid flow functional are satisfied.



a)



b)

Fig. 2. Components of the velocity field in the area of flow development:
a) – longitudinal component for values y=0; 0.25; 0.5;
b) – normal component for values x=0; 0.25; 0.5; 0.75.

We have again come to a "crossroads": we have analytical solutions for two velocity components and two rather than one equation where molecular viscosity is present. Before starting to determine the viscosity function, we should not forget that we should find two solutions that coincide on the surface. The second important point: the equations of conservation of momentum are not equal in value in the boundary layer, because there is a very strong anisotropy of scales, which determines the fact that all components in the second equation of conservation of momentum have negligibly small values compared to the first equation. That is, without violating the assumption of a one-component viscosity function, we should find it from the momentum conservation equation in the horizontal (longitudinal) direction. But, continuing the expansion of existing ideas, let's move away from the dogma that molecular viscosity is a scalar function and assume that in the region of flow development, viscosity has two components, both functions of two coordinates. Then everything falls into place.

As it was just said, we will find two different functions - components of molecular viscosity. Let's start with μ_X. To define this function, we will need the following expressions:

$$\frac{\partial V_X}{\partial y}=-\alpha(1-\exp(\alpha x))\exp(-\alpha y),$$
$$\frac{\partial^2 V_X}{\partial y^2}=\alpha^2(1-\exp(\alpha x))\exp(-\alpha y). \qquad (27)$$

After substituting expression (27) and the others into the first equation (12), we obtain:

$$\frac{\partial \mu_X}{\partial y}-\alpha\mu_X=\exp(\alpha x)\,. \qquad (28)$$

The general solution (28) is the desired *x* - component of the molecular viscosity:

$$\mu_X(x,y)=\left(-\frac{\exp(\alpha(x-y))}{\alpha}+F_1(x)\right)\exp(\alpha y). \qquad (29)$$

Similarly, we find

$$\frac{\partial^2 Vy}{\partial y^2}=-\alpha^2\exp(\alpha x)\exp(-\alpha y).$$

After substituting all the necessary terms and expressions into equation (13), we obtain

$$\frac{\partial\mu_y}{\partial y}-\alpha\mu_y=1-\exp(-\alpha y). \qquad (30)$$

The solution of (30) is the following y -component of the molecular viscosity:

$$\mu_y(x,y)=\exp(\alpha y)F_2(x)+\frac{\exp(-\alpha y)-2}{2\alpha}. \qquad (31)$$

Solutions (29) and (31) cannot coincide at all points of the flow development region. But we can require that they coincide on the surface of the plane, therefore

$$\mu_x(x,y=0)=\mu_y(x,y=0). \qquad (32)$$

Relationship (32) gives a single expression for the two components of molecular viscosity:

$$\mu_x(x,y=0)=\mu_y(x,y=0)=\mu_0(x)=1-\frac{\exp(\alpha x)}{\alpha}. \qquad (33)$$

Expression (33) allows us to draw a completely physical conclusion: since $\alpha>1$, then at the beginning of the area of fluid acceleration (or flow development), the molecular viscosity has a value less than that corresponding to the developed flow and in dimensionless quantities equal to unity. Fig. 3 presents the graph of this function. It can be seen that at the value $\alpha=5$ we have that at the beginning of the flow development region, the viscosity is about 20% lower than the asymptotic value.

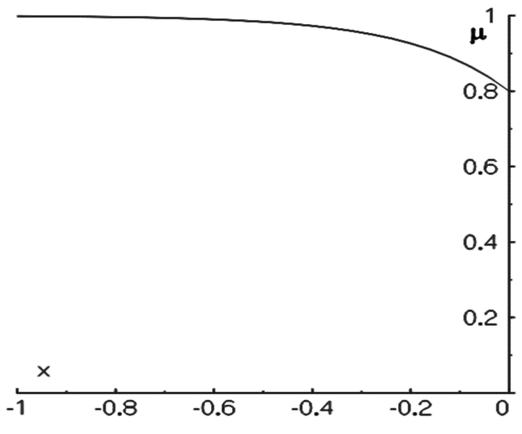


Fig. 3. Molecular viscosity function on the surface of a moving plane

An important characteristic of the flow is tangential stress

$$\tau_{xy}=\tau_{xy}=\mu\left(\frac{\partial V_x}{\partial y}+\frac{\partial V_y}{\partial x}\right)\approx\mu\frac{\partial V_x}{\partial y}.$$

On the surface of the moving plane, we have:

$$\tau_{xy}=\tau_{xy}=-\left(\alpha-\exp(\alpha(x-y))\right)\left(1-\exp(\alpha x)\right). \qquad (34)$$

Therefore, the main component of viscous tangential stresses depends on both coordinates. And, obviously, after passing through the region of flow development due to friction, tangential stresses are constant: even the presence of a vertical coordinate no longer affects this. This is fully consistent with the results of previous studies on an infinite plane [4]. Fig. 4 presents graph (34), which clearly shows the output of the maximum tangential stress function (on the surface of the moving plane) to a constant value. For comparison, in the theory of Blasius [14], as well as that of Stokes and Rayleigh [15, 16], the tangential stress decreases inversely proportional to the square root of the longitudinal coordinate. So, the friction is less and less. When a half-plane moves in a still fluid, the frictional stress, on the contrary, as we have already seen, increases along the flow, and then reaches a constant value. And this is logical.

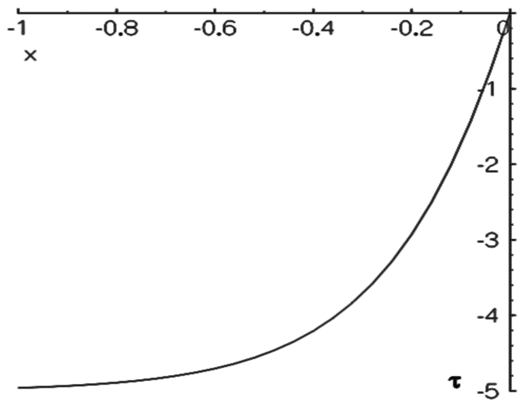


Fig. 4. Maximum tangential stresses: (34), y=0

In conclusion, we will give an expression for the power of the friction force acting on the surface of the plane.

$$P=\tau_{xy}V_x=-\left(1-\frac{\exp(\alpha(x-y))}{\alpha}\right)\times \times\left(1-\exp(\alpha x)\right)^2\exp(-\alpha y). \qquad (35)$$

The graph of the module of maximum values (on the surface of the moving body) of power (35) is presented in Fig. 5. Like all the values given earlier, the power reaches an asymptote - a constant value. In contrast to the solutions [15, 16] obtained on the basis of constancy of molecular viscosity (in the case of an infinite plane) where the power goes to zero in time.

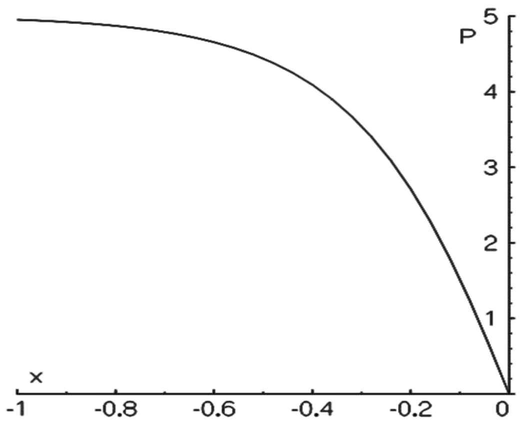


Fig. 5. The maximum power of friction: (34), y =0

Discussion

If you carefully study this work, it will be obvious that the proposed model is not ideal. Thus, the vertical component of the velocity, although obtained from the relevant equations, is not perfectly consistent with the fact that there should be no motion outside the boundary layer. And the resulting solution reaches an asymptote - a constant value. You should not be afraid of this, because the vertical component is several δ times smaller than the longitudinal one. Since δ it is a very small value (less than 0.01), it is clear that velocity measurements at the outer boundary of the boundary layer will simply indicate its small values. But on the other hand, in the theory of the boundary layer, and therefore in this work, the vertical component of the velocity does not play any important role. Therefore, it should be perceived obtained results as a certain approximation to the exact solution. As for the two-component function of molecular viscosity, here too, obviously, not everything is perfect. One could simply ignore the second momentum conservation equation, as Blasius did [14], and then the viscosity would be a function of the coordinates. But deeper considerations led us to the opinion that nothing prevents us in the boundary layer, where the condition of anisotropy is violated, to consider that the viscosity may depend not only on the coordinates, but also on the direction. Therefore, it was decided to leave the second equation of conservation of momentum, but consider that the viscosity function in it is not completely the same as in the first equation. Of course, at the same time, we did not forget the fact that both viscosity functions should coincide on the surface of the body, where the main events take place - the friction of the solid surface and the fluid, which leads to the generation of the boundary layer and two components of the velocity in the area of the flow.

In our previous work [17], with reference to known sources, it was stated that the non-slip condition must be met everywhere on the surface of a solid body, because the flow is viscous. We also criticized the well-known method of discrete vortices, in which a velocity field is actually swept onto the surface of the wing [18, 19]. Now it became clear why exactly this method had a certain success – due to the existing sliding of the fluid on the surface of the solid body in the flow development region.

Conclusions

As research has shown, there are such incompressible fluid flows where the Stokes theory, which postulates the independence of viscosity from spatial coordinates and the mandatory fulfillment of the condition of sticking to a solid surface, cannot adequately describe them. One of such currents is the one generated during the movement of a semi-infinite plane in a stationary liquid. It is in the field of flow establishment that there is a situation where two, according to Stokes, unknown functions (velocity components) correspond to three equations: two conservation of momentum and the third – conservation of mass. Such a situation prompts to look for a way out of the resulting inconsistency. And such a way out, of course, is: to consider molecular viscosity as a function of coordinates, and not as a constant. If for an infinite plane, where there is no region of flow development, this step is already sufficient (see [4, 9]), then in the presence of a region of flow development, one more important step should be taken - to assume that the condition of complete adhesion to the surface of a solid body in the region flow development is not performed. Instead, there is partial sliding, from complete at the beginning of the flow generation region to complete sticking at the end of this region and further downstream. As in previous works [4, 9], optimization methods based on finding the extremum of the fluid flow functional in the boundary layer could not be dispensed with. It was the possibilities of variational calculus that contributed to the finding of the fields of velocity components and molecular viscosity. As is commonly believed, the tangential (principal) stresses corresponding to the normal derivative of the longitudinal component of the velocity are constant after passing through the area of flow development in any cross-section of the boundary layer. Also, together with the strength of the frictional force, the tangential stresses reach an asymptotic constant value at the end of the flow development region, which is logical.

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References

1. Pavlyuchenko A. M., & Shyiko O. M. Kompleksnyy metod rozrakhunku oporu tertya i teploobminu na poverkhni lʹotnykh osesymetrychnykh obʺyektiv pry polʹoti po trayektoriyi z nayavnistyu v prystinnomu prykordonnomu shari neizotermichnosti, styslyvosti, laminarno-turbulentnoho perekhodu ta relaminarizatsiyi [Complex method for calculating the friction resistance and thermal refraction on the surface of flight axisymmetric objects on flight by trajectory with availability in the wall boundary layer non-isothermal, compressive, laminar-turbulent transition and relainarization]. *Aviacijno-kosmicna tehnika i tehnologia – Aerospace technic and technology*, 2018, no. 1, pp. 4-28. DOI: 10.32620/aktt.2018.1.01. (In Ukrainian).

2. Komarov, B., Zinchenko, D., & Andrieiev, O. Vplyv formy kryla na kharakterystyky pry vykorystanni intehrovanoyi rotornoyi sylovoyi ustanovky litaka [Influence of wing shape on characteristics of fan-wing aircraft power plant]. *Aviacijno-kosmicna tehnika i tehnologia – Aerospace technic and technology*, 2023, no. 2, pp. 17-26. DOI: 10.32620/aktt.2023.2.02. (In Ukrainian).

3. Stokes, G. G. On the theories of the internal friction of fluids in motion, and the equilibrium and motion of elastic solids. *Trans. Cambridge Philos. Soc.*, 1845, vol. 8, pp. 287-305.

4. Lukianov, P. V., & Song, L. Optimal character and different nature of flows in laminar boundary layers of incompressible fluid flow. *Problems of friction and wear*, 2022, no. 4(97), pp. 52-60. DOI: 10.18372/0370-2197.4(97)16959.

5. Navier, C. L. M. H. Memoire sur les lois du movement des fluids. *Memoires de l' Academie Royale des Science de l' Institut de France,* 1823, vol. 6, pp. 389-440. (In French).

6. Lauga, E., & Squires, T. M. Brownian motion near a partial-slip boundary: A local probe of no-slip condition. *Physics of Fluids*, 2005, vol. 17, iss. 10, article no. 103102. DOI: 10.1063/1.2083748.

7. Priezjev, N. V. Molecular diffusion and slip boundary conditions at smooth surfaces with periodic and random nanoscale textures. *The Journal of Chemical Physics*, 2011, vol. 135, iss. 20, article no. 204704. DOI: 10.1063/1.3663384.

8. Guo, L., Chen, S., & Robbins, M. O. Effective slip boundary conditions for sinusoidally corrugated surfaces. *Phys. Rev. Fluids*, 2016, vol. 1, iss. 7, article no. 074102. DOI: 10.1103/PhysRevFluids.1.074102.

9. Lukianov, P. V., & Song, L. Unsteady incompressible laminar boundary layer: time and space variable molecular viscosity. *Aviacijno-kosmicna tehnika i tehnologia – Aerospace technic and technology*, 2023, no. 3, pp. 50-60. DOI: 10.32620/aktt.2023.3.06.

10. Kulkarni, Y., Fullana, T., & Zaleski, S. Stream function solutions for some contact line boundary conditions: Navier slip, super slip and generalized Navier boundary conditions. *Proc. R. Soc. A*, 2023, vol. 479, iss. 2278, article no. 20230141. DOI: 10.1098/rspa.2023.0141.

11. Lauga, E., & Stone, H. A. Effective slip in pressure-driven Stokes flow. *Journal of Fluid Mechanics*, 2003, vol. 489, pp. 55-77. DOI: 10.1017/S0022112003004695.

12. Chen, J., Kim, Y. J., & Hwang, W. R. General criteria for estimation of effective slip length over corrugated surfaces. *Microfluids and Nanofluids*, 2020, vol. 24, article no. 68. DOI: 10.1007/s10404-020-02363-1.

13. Cui, H., & Ren, W. A note on the solution to the moving contact line problem with the no-slip boundary condition. *Commun. Math. Sci.*, 2019, vol. 17, iss. 4, pp. 1167-1175. DOI: 10.4310/CMS.2019.v17.n4.a16.

14. Blasius, H. Grenzschichten in Flussigkeiten mit kleiner Reibung. *Zeits. f. Math. u. Phys*, 1908, vol. 56, pp. 1-37. (In German).

15. Stokes, G. G. On the effect of the internal friction of fluids on the motion of pendulums. *Trans. Cambridge Philos. soc.*, 1851, vol. 9, pp. 1-86.

16. Rayleigh, Lord. On the motion of solid bodies through viscous liquids. *Phil. Mag., Ser. VI*, 1911, vol. 21, no. 6, pp. 697-711.

17. Lukianov, P. V., & Song, L. Compact analogs of the models of vortex flows generated by aircraft flight. *Aviacijno-kosmicna tehnika i tehnologia – Aerospace technic and technology*, 2023, no. 3, pp. 4-20. DOI: 10.32620/aktt.2023.5.01.

18. Belotserkovskiy, S. M., & Nisht, M. T. *Otryvnoye i bezotryvnoye obtekaniye tonkikh kryl'yev ideal'noy zhidkost'yu* [Separated and continuous flow of an ideal fluid around thin wings]. Moscow, Nauka Publ., 1978. 352 p. (In Russian).

19. Bulanchuk, G. G., Bulanchuk, O. N., & Dovgiy, S. A. *Metod diskretnykh vikhrevykh ramok so vstavkoy promezhutochnykh tochek na vikhrevoy pelene* [Discrete vortex frame method with insertion of intermediate points on the vortex sheet]. *Visnyk Kharkivs'koho natsional'noho universytetu – Bulletin of the Kharkiv National University*, 2009, no. 863, pp. 37-46. (In Russian).

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ОБЛАСТЬ РОЗВИТКУ ТЕЧІЇ В ПРИМЕЖОВОМУ ШАРІ: ДВОКОМПОНЕНТНА МОЛЕКУЛЯРНА В'ЯЗКІСТЬ ТА ЧАСТКОВЕ КОВЗАННЯ

П. В. Лук'янов, Лінь Сун

Предметом даної роботи є область розвитку ламінарної нестисливої течії рідини в примежовому шарі. Ця течія являє собою приклад, де пряме застосування рівнянь Нав'є-Стокса без-градієнтної ламінарної нестисливої течії рідини, у якому молекулярна в'язкість вважається сталою величиною, що не залежить від просторових координат, призводить до перевизначення математичної моделі. Йдеться про примежовий шар рідини в області встановлення течії в задачі про рух пів-нескінченої площини, де градієнт тиску рівний нулеві. Наявна ситуація, коли кількість рівнянь дорівнює трьом (два рівняння збереження кількості руху і рівняння нерозривності), а кількість невідомих дорівнює двом – числу компонент швидкості. У якості логічного розв'язання отриманої невідповідності запропоновано, як це вже було зроблено для задачі про стаціонарний рух площини і задачі про розгін площини, відійти від хибного твердження про сталість молекулярної в'язкості в без-градієнтному примежовому шарі нестисливої течії та вважати молекулярну в'язкість функцією просторових координат. Необхідність в урахуванні змінного характеру молекулярної в'язкості призвела до відкриття ще одного, другого, недоліку теорії Нав'є-Стокса. Цей нетривіальний недолік було виявлено під час застосування оригінального чисельно-аналітичного методу розв'язання задачі про течію рідини в області встановлення руху. Як відомо, рівняння Нав'є-Стокса доповнюються граничними умовами. Найважливішою умовою є умова прилипання (не ковзання) рідини на поверхні твердого тіла, яка, до речі, не випливає ні з якого фізичного закону. В результаті на поверхні пів-площини (або тіла), що рухається, складова швидкості, що співпадає із напрямком руху, має стале значення, яке дорівнює швидкості тіла. З рівняння нерозривності одразу випливає, що нормальна похідна від нормальної компоненти швидкості повинна бути рівною нулеві уздовж поверхні площини (тіла), так як повздовжня похідна від повздовжньої компоненти швидкості перетворюється на нуль. Однак, цілком очевидно, що нормальна до поверхні площини (тіла) компонента швидкості змінюється поперек примежового шару – в області розвитку течії, що означає наявність нормального градієнту (обох складових) швидкості. Конфлікт, або протиріччя, долається шляхом відходу від загальноприйнятої умови прилипання до умови часткового прилипання, або по суті наявності ковзання. Так як і при різкому гальмуванні будь якого транспортного засобу повна зупинка відбувається не миттєво, а за деякий скінчений час и шлях, то і в випадку руху тіла в нерухомій рідині відбувається не миттєве прилипання, а поступове – від повного ковзання, коли частинка рідини щойно зустрілася із рухомою площиною, до повного прилипання наприкінці (і далі) області розвитку течії. **Методи досліджень.** В роботі використовуються суто теоретичні методи, що базуються на використанні варіаційного числення, законів фізики та уявлень із повсякденного життя. **Висновки.** Вдосконалена модель в'язкої н'ютонівської рідини в області розвитку течії в примежовому шарі. На підставі припущень про змінний характер молекулярної в'язкості, яка вже має дві складові, та відходу від виконання умови прилипання, отримані аналітичні розв'язки для обох складових швидкості, а також обох складових молекулярної в'язкості. Наведено порівняння отриманих результатів із результатами інших робіт.

Ключові слова: літак; вертоліт; область розвитку течії; примежовий шар; часткове ковзання.

Лук'янов Павло Володимирович – канд. фіз.-мат. наук, старш. наук. співроб., доц. каф. гідро газових систем, Національний авіаційний університет, Київ, Україна.

Лінь Сун – магістр, асп. каф. гідрогазових систем, Національний авіаційний університет, Київ, Україна.

Pavlo Lukianov – Candidate of Physics and Mathematics Sciences, Senior Researcher, Associate Professor at Hydro-Gas Systems Department, National Aviation University, Kyiv, Ukraine,
e-mail: Pavlo.Lukianov@npp.nau.edu.ua, ORCID: 0000-0002-5043-6182.

Lin Song – Master of Technical Sciences, PhD at Hydro-Gas Systems Department, National Aviation University, Kyiv, Ukraine,
e-mail: 592123500@qq.com, ORCID: 0000-0001-6328-4889.