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COMPACT ANALOGS OF THE MODELS OF VORTEX FLOWS GENERATED BY AIRCRAFT FLIGHT

The subject *of this work is the development of compact analogs of vortex flows models, which are used in the modeling of vortex structures observed during the flight of an aircraft and the motion of a body in a fluid. In particular, two significant misunderstandings prevailing in this area of science are highlighted. The first misunderstanding is that the stationary motion of fluid parcels in a circle is treated as an inviscid vortex. Therefore, any vortex flow model that does not explicitly contain viscosity is considered to describe inviscid vortex motion. It has been proven that this is not so: the stationary viscous motion of fluid parcels in circular orbits corresponds to the self-balance of one force - the force of viscosity. This conclusion, in an explicit form, was made for the first time. This is very important because it changes our ideas about force balance, where two or more forces of different natures must necessarily be present. Overcoming this misunderstanding opens the way for creating compact analogs of existing models of vortex motions. Along the way, one more - the second general misunderstanding in the field of vortex dynamics was eliminated. Wherever we read it, we can see that the compactness of the vortex flow is associated with the compactness of the vorticity field. This is facilitated by the fact that the equations for vorticity and not for velocity are considered. As a result, except for one or two models of vortices, which correspond to the rotation of the entire space, up to infinity, this violates the fundamental law of physics - the law of conservation and transformation of energy. It is about the fact that, as a second misunderstanding, an error is assumed when calculating the kinetic energy of the vortex flows: the Jacobian in cylindrical (polar) coordinates is not considered. As a result, all the mentioned models of vortex flows, which correspond to the hyperbolic law as their asymptotics in the periphery, have infinite kinetic energy. Certainly, this does not correspond to the formation and evolution of compact vortex structures. Therefore, in this work, based on overcoming the aforementioned misunderstandings, many previously obtained models of compact vortex flows, as well as those obtained for the first time, are presented. In particular, this applies to the turbulent vortex flow during the formation of a vortex sheet, which is a compact analog of the Burgers-Rott vortex - both the classical one corresponding to laminar motion and the one consisting of a laminar flow in the core and a turbulent flow on the periphery of the vortex.* ***Research methods*** *are entirely theoretical. Well-known theorems of theoretical mechanics, mathematical theory of field, and calculus of variations, etc. are used. The obtained solutions are compared with the existing corresponding analogs of non-compact flows.* ***Conclusions.*** *Using the methods of calculus of variation, it was possible to show the possibility of the formation of quasi-solid-like rotational motion in a boundary layer of an incompressible fluid. The very presence of viscosity, or rather its taking into account (boundary layer), indicates a possible transition of the flow from plane-parallel motion to the just-mentioned rotational one due to the Kelvin-Helmholtz instability. In addition, two new models of the Burgers-Rott vortex flow were obtained in this study. The first model uses the general solution obtained by Burgers, but this model corresponds to a combined vortex: although the velocity field is continuous, the vorticity field has a discontinuity - at the point of maximum velocity. It is proved that such an approach is quite possible: the equation of motion is satisfied everywhere, i.e., at every point in space, and the tangential stresses are continuous functions. Since the periphery of the Burgers-Rott vortex is an unstable flow, another model is proposed - with a laminar core and a turbulent periphery. Certainly, the motion of fluid parcels in the peripheral region is described by a velocity distribution other than that of Burgers. Finally, the possible use of known models of compact vortex flows to simulate the von Karman vortex street is considered, with an indication of the advantages of these models.*

Keywords*: aircraft; vortex flows; Burgers-Rott vortex; vortex sheet; Karman vortex street; two misuderstandings in vortex dynamics.*

Introduction

Of all the physical fundamental laws of conservation in mechanics, conservation of momentum occupies a special place. Regardless of the nature of the interaction (elastic or inelastic collision), this law is always valid for the total momentum of the interacting (both discrete and distributed) system of bodies. In connection with what has just been said, it is not by chance that the inertial component of motion is the main one in the creation of lifting force. Pushing air down from themselves, birds and insects are able to hover in the air thanks to the law of conservation of momentum. The aircraft, as you know, is not capable of flapping its wings. To create lifting force, aircraft needs to have a component of the velocity of rectilinear motion. Being tilted at some angle (attack) in relation to the movement of the aircraft, the wing, like birds and insects, repels the oncoming air flow from itself down and, thus, creates the main (inertial) component lifting force [1].

When a body moves in the fluid, in the immediate vicinity of its surface a boundary layer is formed - a flow region in which the velocity changes from the maximum, equal velocity of the body's motion at this point in space, to an almost zero value, 100-1000 times less than the maximum. In the early period of its development, when the theory of viscous fluid was not yet developed or accepted, in hydromechanics the boundary layer was interpreted as a mass attached to the body [2]. So, in order to get closer to a real description of the phenomenon, it was proposed to consider, on the basis of the model of inviscid fluid flow, the motion of a body together with some volume of fluid around it. It is quite obvious that this approach to describing the motion of an airplane is not suitable - at each subsequent moment of time, the aircraft, during its motion, interacts with new and new air masses.

As you know, the flight and the interaction of the wing with the wind causes the generation of vortex motions. In particular, this applies to the detachment of the vortex from the surface of the wings and other parts of the aircraft [3], as well as the formation of a vortex flow in the wake behind the aircraft [4]. One of the first systematizations of the physics of vortex formation was made in [5]. Works [6, 7] are devoted to the study of the vortex sheet. For a comprehensive understanding of vortex flow models, one should refer to the monograph [8]. The nature of the creation of any vortex flow is the presence of a velocity shift. This conclusion follows directly from the general formula (definition) of vorticity as a mathematical operator of the rotor of the velocity vector: the absence of gradients (shift) of velocity along spatial coordinates cannot create a vortex. But that's not all: even a potential flow in circular orbits (a point vortex) is a viscous motion, and those other vortex flows, which are considered inviscid due to the obvious lack of viscosity in the solution, are nothing another, as a viscous stationary motion, in which the forces of viscous tangential stresses balance themselves [9].

In his speech to the Royal Society of Great Britain on the occasion of his awarding of the gold medal, Ludwig Prandtl described in detail all the knowledge known to him at that time about the vortical nature of a body being flown around a by stream of fluid impinging on it (reverse problem) [5]. According to Prandtl, vorticity is not generated anywhere, but in the boundary layer, the nature of which is the presence of viscosity in shear flows. In his report, Prandtl presented methods of prevanting the separation of vortices in the boundary layer, which are used in modern research [10]. However, like everyone else, Prandtl mistakenly considered the reverseability of the direct and inverse problems - the motion of a body in a fluid and the flow of a fluid past a immobile body. This, as it became known recently [11, 12], is incorrect.

In shear flows, at the interface of two media, the Kelvin-Helmholtz instability may appear. One of the early works is the article by Meyron et al. [13], which is devoted to the analytical structure of the vortex sheet. The study of the formation of features (singularities) at the early stage of the nucleation of the vortex sheet is given attention in the paper [14]. The works [15, 16] are devoted to the testing and modeling of the Kelvin-Helmholtz instability, and [17, 18] to the study of the nonlinear properties of this instability.

A study of the stability of the unsteady Kelvin-Helmholtz flow can be found in [19]. In vortex dynamics, there is such a concept as self-organization - the formation of certain vortex structures from chaotic motion, which are called coherent or long-lived. From this point of view, the two-dimensional Kelvin-Helmholtz instability is considered in [20]. Various initial conditions can be used to control the boundary layer and the Kelvin-Helmholtz instability. The work [21] is devoted to the influence of the initial conditions on the further development of the Kelvin-Helmholtz instability. This instability leads to the formation of a vortex sheet, which makes a significant contribution to the formation of the lifting force of the wing [5].

Despite the specified nature of vortex formation due to the presence of viscosity, methods based on the use of Euler's equations, i.e., those in which viscosity is not taken into account, have been widely used for calculating the lift force (and resistance). One of them is the so-called *Method of discrete vortices*, which is described in monographs [22, 23] and article [24]. Further development of this method and its application to applied problems is contained in [25, 26]. This method ignores the presence of a fluid boundary layer on the surface of the body and considers that the fluid flows around the body, the magnitude and direction of the velocity of which is modeled by a pair of oppositely directed vortices - a dipole (see also work [27]). In order to take into acount the structure of the boundary layer, one should consider (model) a viscous vortex flow a priori. Fortunately, over the previous 10-15 years, there has been a clear awareness of the shortcomings of existing vortex flow models and, as a result, a more accurate description of them. The starting point was the reluctance to put up with the fact that natural vortices have a compact structure and, instead, they are put in line with current models where the entire space rotates - up to infinity. This is how the first work [28] appeared. But the point vortex model, on the basis of which the above-mentioned method of discrete vortices was created, still reigned supreme. But it was replaced by a compact analogue – the model of a quasi-point vortex [29], the main advantage of which is the ability to specify any finite size of the vortex, which is close to the classic – point vortex.

Since the vast majority of fluid flows are turbulent, the model of a quasi-point (laminar) vortex was generalized to the case of a turbulent flow - under certain restrictions [30, 31]. A little later, it became clear how on the basis of one model it is possible to obtain, through limit transitions, all existing basic models - point vortex, quasi-point vortex and Rankin vortex. Therefore, this model was called the universal compact vortex model [32]. The theoretical justification, based on classical theorems, of the possibility of the existence of compact vortex flows came a little later - after numerous discussions and speeches at seminars of leading institutes of the National Academy of Sciences of Ukraine [33].

Finally, since the notion of inertial stability in vortex motion is extremely important, because it indicates whether the flow is laminar or turbulent, a study was conducted, the results of which are contained in [34]. Awareness of the need to take into account the variability of viscosity in the boundary layer [11, 12] made it possible to look at the analyzed motion in a new way: since the vortex sheet is formed in the boundary layer, in the vortex model, viscosity has the right to be a variable value - inside the boundary layer, of course. The just mentioned ideas served as the generator of this work. Later, during a comprehensive study of the problem of vortex formation during the motion of a body in a fluid, and an airplane in particular [4], it became clear that one should not limit oneself to a single vortex sheet - one should also pay attention to free vortex flows, such as the Burgers-Rott vortex [35, 36] and the von Kármán vortex street.

Further presentation is structured as follows: the problems and purposes of this work are formulated, general misunderstandings in the theory of vortex motions are clarified, in particular, the identification of viscous coherent (long-lived) vortices with inviscid ones and modeling of compact free vortex flows using velocity distributions, kinetic energy in which is equal to infinity, which is obviously impossible due to the law of conservation of energy. Next, the models of the vortex sheet, the descending free vortex, and Karman vortex street are considered. It should be noted that the concept of a mathematical model is also being expanded [37]: while remaining in the class of continuous functions for the velocity field, which is important for the continuity of the pressure field, it is possible to use piecewise continuous distributions for the vorticity field while maintaining generality. At the same time, the equation describing the motion remains valid (is fulfilled) at all points of the vortex flow domain.

1. Problem formulation

The purpose of the work to describe, with the help of models of compact vortices corresponding to a viscous fluid flow:

– motion in a vortex sheet;

– free descended vortex flow, described by a Burgers-Rott vortex, as well as a von Kármán vortex street, which is also described by models of inviscid vortices (by with the exception of [38]);

– in the course of solving the problem, in addition to using already known models, obtain new models of compact vortex flows generated during aicraft flight.

2. Two serious misunderstandings as for models of vortex flows

Misunderstanding 1. *It is treated that if the vortex flow is described by an equation or a system of equations where molecular viscosity is absent in an explicit form, then the flow is inviscid. At the same time, intuitively, viscosity is considered unambiguously as a source of nonstationarity of the flow, which leads to the diffusion of vorticity.*

This misunderstanding is very serious – not trivial. From a formally mathematical point of view, when there is no viscosity in the solution, we claim that there is no vortex diffusion mechanism and thus the vortex flow is inviscid. But few could guess that such a state corresponds to the stationary flow of a viscous fluid along circular trajectories. Batchelor [9] showed that the force of viscosity in a fluid can be in self-equilibrium. How! - everyone will exclaim. After all, balance means the presence of two or more forces. That is why it is a balance - to equate one force with another. However, a balance of one force is possible. Given its exceptional importance for understanding the physics of the phenomenon, we will show, following Batchelor, how this balance occurs.

One of the main theorems of theoretical mechanics is the theorem on the change in the angular momentum of a body during its rotation. Let $\vec{L}$ be the angular momentum and the momentum of external forces $\vec{M}_O$ applied to the body be:

$$\vec{L}=m\cdot\vec{r}\times\vec{V}, \quad \vec{M}_O=\vec{M}_O\left(\vec{R}^e\right).$$

Then the theorem on the change of the angular momentum is described by the following equality [39]:

$$\frac{d\vec{L}o}{dt}=\vec{M}o\left(\vec{R}^e\right). \tag{1}$$

If we now imagine the motion of fluid parcels along circular trajectories (the domain along the axis of rotation is infinite), then the internal friction in the fluid is described by a single component of the viscous stress tensor

$$\sigma_{r\theta}=\mu\left(\frac{\partial V_\theta}{\partial r}-\frac{V_\theta}{r}\right). \qquad (2)$$

The momentum of external forces applied to an infinitely thin ring cross-section domain is equal to [9]

$$M_z=2\pi\mu r^2\left(\frac{\partial V_\theta}{\partial r}-\frac{V_\theta}{r}\right). \qquad (3)$$

At the same time, the angular momentum is determined by the formula [9]

$$L_z=2\pi r\rho\cdot r\cdot V_\theta. \qquad (4)$$

Substituting (3) and (4) into (1), we obtain

$$\frac{\partial\left(2\pi\rho r^2 V_\theta\right)}{\partial t}=\frac{\partial}{\partial r}\left\{2\pi\mu r^2\left(\frac{\partial V_\theta}{\partial r}-\frac{V_\theta}{r}\right)\right\}, \qquad (5)$$

or, after calculating the derivatives in the right and left parts of (5), we obtain the well-known Navier-Stokes equation in the Gromeka-Lamb form [9, 40]

$$\frac{\partial V_\theta}{\partial t}=\nu\left(\frac{\partial^2 V_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial V_\theta}{\partial r}-\frac{V_\theta}{r^2}\right). \qquad (6)$$

The long-awaited conclusion follows from (1) and (6): during steady-state viscous motion of fluid parcels in circular orbits, the main vector of external forces applied to fluid cylindrical surfaces is zero. This becomes possible thanks to the self-balance of viscous friction in the fluid. And by no means refers to the lack of viscosity. The mistake of all those who believe this inviscid flow is explained by the stationary analog (6), which has the form

$$0=\left(\frac{\partial^2 V_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial V_\theta}{\partial r}-\frac{V_\theta}{r^2}\right). \qquad (7)$$

Yes, indeed, there is no viscosity in (7) – in an explicit form. But it corresponds to nothing else but viscous the self-balance. That is why the fundamental solution of (7) has the form

$$V_\theta=C_1 r+C_2 r^{-1} \qquad (8)$$

that is widely used in vortex dynamics. To close the issue of viscosity, we will show that the point vortex and Rankine vortex models correspond to viscous flows. Indeed, the distribution of the velocity field in a point vortex is described by the expression [8]

$$V_\theta=\frac{\Gamma}{2\pi r}. \qquad (9)$$

Let's substitute (9) in (2). We then obtain that

$$\sigma_{r\theta}=\mu\frac{\Gamma}{2\pi}\left(-\frac{1}{r^2}-\frac{1}{r^2}\right)=-\mu\frac{\Gamma}{\pi r^2}\neq 0. \qquad (10)$$

Thus, in its entire flow domain, a point vortex has non -zero viscosity stresses. Since a Rankine vortex consisting of a core $0\le r\le a$ and a periphery $r\ge a$ has a velocity distribution of the form

$$V_\theta=\begin{cases}\dfrac{V_0 r}{a}, & 0\le r\le a,\\ \dfrac{V_0 a}{r}, & r>a.\end{cases} \qquad (11)$$

then the entire peripheral region, according to (10), is a viscous flow.

Misunderstanding 2. *Second misunderstanding is also present everywhere. This is the identification of the compactness of the vorticity field with the compactness of the velocity field, that is, of the vortex flowt itself. As a result, practically all models of vortex flows, having a compact vorticity field, have the same unnatural property: they cover an infinite domain of space with their rotation and have infinite kinetic energy. It is obvious that the generation of the vortex in a finite time is due to the finite power (instability) is by no means capable of creating an object with infinite kinetic energy.*

We will show that the kinetic energy in a point vortex, and at the same time in any other in which the velocity field tends asymptotically to such a distribution, is equal to infinity. Indeed, according to the definition of kinetic energy and expression (9), we have:

$$E_k=\rho 2\pi\frac{1}{2}\int_a^r\left(\frac{\Gamma}{2\pi r}\right)^2 r\,dr,$$

$$E_k=\rho\frac{\Gamma^2}{4\pi}\ln r\Big|_a^r\to\infty,\quad r\to\infty. \qquad (12)$$

What does almost everyone do? They forget to multiply by the Jacobian r when transition to a cylindrical (or polar) coordinate system. And instead of (12) they obtain

$$E_k = \frac{\rho}{2}\int_a^{\infty}\left(\frac{\Gamma}{2\pi r}\right)^2 dr = \frac{\rho}{2}\left(\frac{\Gamma}{2\pi}\right)^2 \frac{1}{r}\Big|_a^{\infty} = = -\frac{\rho}{2}\left(\frac{\Gamma}{2\pi}\right)^2 \frac{1}{a}\Big| < 0,\ r \to \infty. \quad (13)$$

It turns out "just fine": according to (13), the kinetic energy of the periphery of any vortex, which has the asymptote of a point vortex, has a negative value (the initial one at the boundary of the vortex core has a minus sign). So, complete nonsense. How can the kinetic energy of any part of a vortex, as well as any body of a system of bodies, have a negative value? This issue is not covered in the literature. Unfortunately, and why it was necessary to dwell on this issue in such detail, at present, the Rankine vortex and the point vortex are considered inviscid flows [41] with finite kinetic energy.

3. Vortex sheet and its simulation

When modeling the vortex sheet as a system of dipoles, it is assumed that the flow is inviscid. Indeed, each of the point vortices (9) satisfies the stationary Navier-Stokes equation in the form of Gromecka-Lamb (7). Since within the framework of the constant molecular viscosity model equation (7) is linear, it is assumed that the superposition of these solutions is also a solution of (7).

The correct understanding of the physics of the studied phenomenon is constantly hindered by the error that the direct and inverse problems are completely reversible. If a viscous fluid (air) flows past immobile body then near each point of the body's surface, the velocity of the fluid, generally speaking, is different due to the presence of flow development region. The value of this velocity, along with the direction of the flow, is modeled by a pair of elementary vortices [22]. And the vortex sheetis considered (modeled) as a singular field of vorticity capable of generating a saltus of the velocity vector of finite magnitude. Recall that the velocity fields inside the gradient and gradient-free boundary layers are described by different functions, that is, they are different [11]. And the main thing - when a body moves in a fluid (flight), the molecular viscosity inside the gradient-free boundary layer is variable. Since it is important to define the "initial conditions" as accurately as possible for the use of modern design and calculation systems, the structure of the vortex sheet at the stage of its formation is of great importance. In this regard, it should be noted the works [42, 43], which in particular indicate three types of vortex structures formed in turbulent flows: the core of the Burgers vortex (the periphery is not described by this vortex), structures similar to a curved vortex layer and a flat vortex sheet.

We will try to answer, based on the available information, the main question: how, in what way is the vortex sheet formed? We will assume that the formation (generation) of the vortex sheet occurs in the boundary layer. Since, as noted, any vortex flow is shear, the presence of a boundary layer contributes to the formation of vortices. The physical nature of the formation of vortices is the Kelvin-Helmholtz instability. You can give the simplest analogy with theoretical mechanics. If the outer boundary of the gradient-free boundary layer (direct problem) is identified with the ground, and the speed of the surface of the moving body is identified with the speed of the wheel at the top point (double the speed of motion of the wheeled vehicle), then we obtain that a vortex, like a wheel, rolls along the outer still (approximately) boundary at the boundary layer. It should be noted that this representation is, in fact, similar to Milyonshchikov 's hypothesis [44], which for the inverse problem (flow along a still surface) suggest that turbulent vortices seem to roll along the surface of the body. Such a hypothesis allowed Milyonshchikov to obtain the well-known logarithmic law of wall for a turbulent flow. Since the translational speed of the wheel is half the maximum (relative to the instantaneous center of rotation - the point of contact of the wheel with the ground), then, by analogy, the vortices formed in the vortex sheet will rolling downstream from the body (wing) with a finite speed (perhaps equal to half the speed of moving body).

Since the discovery of different structures of gradient and gradient-free boundary layers of incompressible fluid flows [11] is based on the variational principles of mechanics, it is logical to use them here as well - to describe the nature of the vortex sheet formation. In order to correctly choose the appropriate functional, we recall that Prandtl also indicated the turbulent nature of the vortex sheet [5]. We now substantiate his statement on the basis of the calculus of variations. Omitting the details, we note that the vortex nature of the flow in the boundary layer requires us to consider not the extreme of the fluid flow rate through the cross-section of the boundary layer [11], but the extreme (maximum) of the rotor velocity vector

$$J = \int_{\Omega} \operatorname{rot}\vec{V} d\Omega \to \text{extr} . \quad (14)$$

This extreme, generally speaking, is conditional, because we must add to (14) the corresponding Navier-Stokes equation for V_θ and take into account the variable nature of $\mu(r)$. As we consider

$$\vec{V} = \left(V_r = 0, V_\theta(r), V_z = 0\right), \quad (15)$$

then, according to (15), the only component of the velocity rotor is

$$\mathrm{rot}_z\vec{V}=\frac{1}{r}\frac{\partial}{\partial r}\left(rV_\theta\right). \quad (16)$$

Condition (14), according to (16), acquires the explicit form

$$J=\int_0^{2\pi}\int_{-\delta/2}^{\delta/2}\frac{1}{r}\frac{\partial}{\partial r}\left(rV_\theta\right)rdr \to \text{extr.} \quad (17)$$

If we consider the usual extreme (17), then the corresponding Euler equation turns into the identity. So let's consider a conditional extreme. The corresponding Lagrange function has the form:

$$\Phi=V_\theta+r\frac{dV_\theta}{dr}+\lambda\left[\left(\frac{d\mu}{dr}+\frac{2\mu}{r}\right)\left(\frac{dV_\theta}{dr}-\frac{V_\theta}{r}\right)\right]+ +\lambda\mu\left(\frac{d^2V_\theta}{dr^2}-\frac{1}{r}\frac{dV_\theta}{dr}+\frac{V_\theta}{r}\right). \quad (18)$$

Since Φ contains two unknown functions, we have two additional Euler-Lagrange (calculus of variations) equations at our disposal. They are

$$\begin{cases} 1+\lambda\left(-\frac{1}{r}\left(\frac{d\mu}{dr}+\frac{2\mu}{r}\right)\right)-\frac{d}{dr}\left(r+\lambda\left(\frac{d\mu}{dr}+\frac{2\mu}{r}\right)\right)+ \\ \quad +\frac{d^2}{dr^2}\lambda\mu=0, \\ \left(\frac{dV_\theta}{dr}-\frac{V_\theta}{r}\right)\left(\frac{2\lambda}{r}-\frac{d\lambda}{dr}\right)=0. \end{cases} \quad (19)$$

From the second equation of system (19), it follows that the Lagrange multiplier has the form:

$$\lambda=C\cdot r^2. \quad (20)$$

It is easy to check that substituting (20) into the first equation of system (19) turns it into an identity. The following conclusion can be drawn from the above: within the framework of the variable molecular viscosity model within the boundary layer, the vortex motion (15) automatically delivers the conditional extreme of the vorticity functional (17). Therefore, whatever velocity distribution in the framework of (15) we set in advance, it will meet the necessary conditions of the conditional extreme for the vorticity functional. This conclusion gives us the opportunity to use different models of viscous stationary vortex flows that are the solutions of equation (19). On the other hand, we have not been able to determine exactly which velocity distribution is natural for the vortex sheet formed in the boundary layer. The standard trick is to complicate the model. Let's change from the Navier-Stokes equations with variable molecular viscosity for the corresponding Reynolds equations (turbulent fluid motion)

$$0=\left(\frac{d\mu}{dr}+\frac{2\mu}{r}\right)\left(\frac{d\bar{V}_\theta}{dr}-\frac{\bar{V}_\theta}{r}\right)+\mu\left(\frac{d^2\bar{V}_\theta}{dr^2}-\frac{1}{r}\frac{d\bar{V}_\theta}{dr}+\frac{\bar{V}_\theta}{r}\right)+ +\frac{d}{dr}\left(A_\theta\frac{d\bar{V}_\theta}{dr}\right)+\frac{2}{r}\frac{d\bar{V}_\theta}{dr}. \quad (21)$$

Having carried out similar procedures for finding the conditional extreme, we will also obtain the expression (20) and additionally the equation

$$-Cr^2\frac{d^2\bar{V}_\theta}{dr^2}=0, \quad (22)$$

the solution of which is

$$\bar{V}_\theta=C_1r+C_2. \quad (23)$$

So, after considering the general case of turbulent motion , we obtained as a result that a moving plane can generate a quasi-solid rotational motion (see formula (23)). This means that when the transition to unstable (turbulent) type of motion, boundary layer can turn into a system of vortices, the centers of which move (downstream) with a speed equal to half the maximum speed relative to the solid surface. At the same time, we do not forget that molecular viscosity and turbulent viscosity are functions of the distance to the surface of the body. Next, the vortex sheet detaches from the surface and the vortex flow becomes compensated - the integral of the vorticity over the entire domain of the vortex is equal to zero. This is a necessary and sufficient condition that the vortex is isolated - it occupies a finite domain of space. This process is called *shedding* [45, 46]: a peripheral domain of vorticity of the opposite sign is formed around the main (core) domain of vorticity of one sign. It is this, the peripheral domain, that allows the vortex to have a finite size - at any time moment after detaching. The effect of distance to boundary on Kelvin-Helmholz instability studied in [47].

4. Descending vortex: the Burgers-Rott model and its compact counterpart

After the separation of the vortex sheet, a so-called desending vortex is formed. This vortex is approximated

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by the Burgers-Rott vortex model [35, 36]. The Burgers-Rott vortex is an exact solution of the stationary Navier-Stokes equations. The following velocity field is understood under this model

$$\begin{cases} V_r = -\alpha \cdot r, \\ V_z = 2 \cdot \alpha \cdot z, \\ V_\theta = \dfrac{\Gamma}{2\pi r} \cdot g(r). \end{cases} \quad (24)$$

If, instead of the third equation (24), that is, the expression for the azimuthal velocity, we just write

$$V_\theta = V_\theta(r)\,, \quad (25)$$

and substitute (25) and the first two equations (24) into the following system of Navier-Stokes equations (in cylindrical coordinates)

$$\rho\left(\frac{\partial V_r}{\partial t} + \bar{V}_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + \bar{V}_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}\right) =$$
$$= -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2} + \frac{\partial^2 V_r}{\partial z^2} - \right.$$
$$\left. - \frac{V_r}{r^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta}\right),$$
$$\rho\left(\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} - \frac{V_r V_\theta}{r^2}\right) =$$
$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{\partial^2 V_\theta}{z^2} - \right.$$
$$\left. - \frac{V_r}{r^2} + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta}\right),$$
$$\rho\left(\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}\right) = -\frac{\partial p}{\partial z} +$$
$$+ \mu\left(\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial Vz}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2} + \frac{\partial^2 V_z}{\partial z^2}\right), \quad (26)$$

$$\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(rV_\theta) + \frac{\partial}{\partial z}(rV_z) = 0, \quad (27)$$

then we obtain:

$$\alpha^2 r - \frac{V_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r},$$
$$\frac{d^2 V_\theta}{dr^2} + \left(\frac{1}{r} + \frac{\alpha}{\nu} r\right)\frac{dV_\theta}{dr} + \left(\frac{\alpha}{\nu} + \frac{1}{r^2}\right)V_\theta = 0\,, \quad (28)$$
$$4\alpha^2 z = -\frac{1}{\rho}\frac{\partial p}{\partial z}\,.$$

The general solution of the equation (28) will be precisely the Burgers vortex

$$V_\theta = \frac{1}{r}\left(C_1 + C_2 \exp\left(-\frac{1}{2}\Lambda r^2\right)\right). \quad (29)$$

In (29)

$$\Lambda = \frac{\alpha}{\nu}\,. \quad (30)$$

The parameter Λ indicates the physics of the process - the balance between viscosity and advection in the radial direction. Vorticity field in such a vortex is compact according to Saffman [7]

$$\omega_z = -\Lambda \cdot C_2 \cdot \exp\left(-\frac{1}{2}\Lambda r^2\right). \quad (31)$$

However, the velocity field is not compact. According to (12), the kinetic energy in the Burgers-Rott vortex is equal to infinity, which contradicts the law of conservation of energy. To overcome this problem (inconsistency with the energy conservation law), let's transform (29) into a compact vortex. We made an attempt to use the representation based on a quasi-point (compact) vortex instead of the third relation (24) [28]

$$V_\theta = \frac{\Gamma}{2\pi r}\left(1 - \left(\frac{r}{R_V}\right)^2\right) g(r)\,,$$

where R_V is a vortex radius that is valid further for other formulars as well.

As a result, the solution (29) is obtained. So what to do to solve this problem? The answer is that for this we need to depart from the usual notions that not only velocity but also its derivative must be continuous functions. Fortunately for us, that was exactly what it was about during the discussion of the modeling of the vortex sheet : there the vortex field has a saltus. The fact is that from a physical point of view, equation (7) holds not only for velocity distributions given by one function, but also for any finite number of piecewise continuous distributions constructed on the basis of (29). In other words, like the composed Rankine vortex (11), we can consider the Burgers-Rott vortex also as a composed, the field of the azimuthal component of the velocity in which is given by the relations:

$$V_\theta = \begin{cases} \dfrac{\Gamma}{2\pi r}\left(1-\exp\left(-\dfrac{1}{2}\Lambda r^2\right)\right), \ 0 \le r \le r_k; \\ \dfrac{\Gamma}{2\pi r}\left(C_3 + C_4\exp\left(-\dfrac{1}{2}\Lambda r^2\right)\right), \ r_k < r \le R_V, \end{cases} \quad (32)$$

where r_k is a vortex core radius.

Constants C_3, C_4 are determined under the following conditions: continuity of the velocity field

$$C_3 + C_4\exp\left(-\frac{1}{2}\Lambda r_k^2\right) = \frac{\Gamma}{2\pi r}\left(1-\exp\left(-\frac{1}{2}\Lambda r_k^2\right)\right)$$

and the condition for the compactness of thevelocity field (the vortex is isolated)

$$0 = \frac{1}{R_V}\left(C_3 + C_4\exp\left(-\frac{1}{2}\Lambda R_V^2\right)\right).$$

After solving the system of equations with respect to C_3, C_4 we obtain a desired model that is *compact analog for Burgers-Rott vortex*

$$V_\theta = \begin{cases} \dfrac{\Gamma}{2\pi r}\left(1-\exp\left(-\dfrac{1}{2}\Lambda r^2\right)\right), \ 0 \le r \le r_k; \\ \dfrac{\Gamma}{2\pi r}\left(\dfrac{1-\exp\left(-\frac{1}{2}\Lambda r_k^2\right)}{\exp\left(-\frac{1}{2}\Lambda R_V^2\right)-\exp\left(-\frac{1}{2}\Lambda r_k^2\right)} \times \left[\exp\left(-\dfrac{1}{2}\Lambda R_V^2\right)-\exp\left(-\dfrac{1}{2}\Lambda r^2\right)\right]\right), \ r_k < r \le R_V. \end{cases}$$

(33)

The vorticity field, according to (33), has the following distribution

$$\omega_z = \begin{cases} \Lambda\dfrac{\Gamma}{2\pi}\exp\left(-\dfrac{1}{2}\Lambda r^2\right), \ 0 \le r \le r_k; \\ \Lambda\dfrac{\Gamma}{2\pi}\left(\dfrac{\left(1-\exp\left(-\frac{1}{2}\Lambda r_k^2\right)\right)\exp\left(-\frac{1}{2}\Lambda r^2\right)}{\exp\left(-\frac{1}{2}\Lambda R_V^2\right)-\exp\left(-\frac{1}{2}\Lambda r_k^2\right)}\right), \ r_k < r \le R_V. \end{cases}$$

(34)

One can see at Fig. 1 velocity and vortixity distributions of Burgers-Rott vortex and just obtained it's compact counterpart. According to velocity distribution (33) all the space is rotating. In return, the solution (29) shows that the rotation is finite space domain (approximately 3 non-dimensional units in radial direction). Vorticity distribution has two domains of opposite sign that make it possible the existance of vortex flow in compact domain.

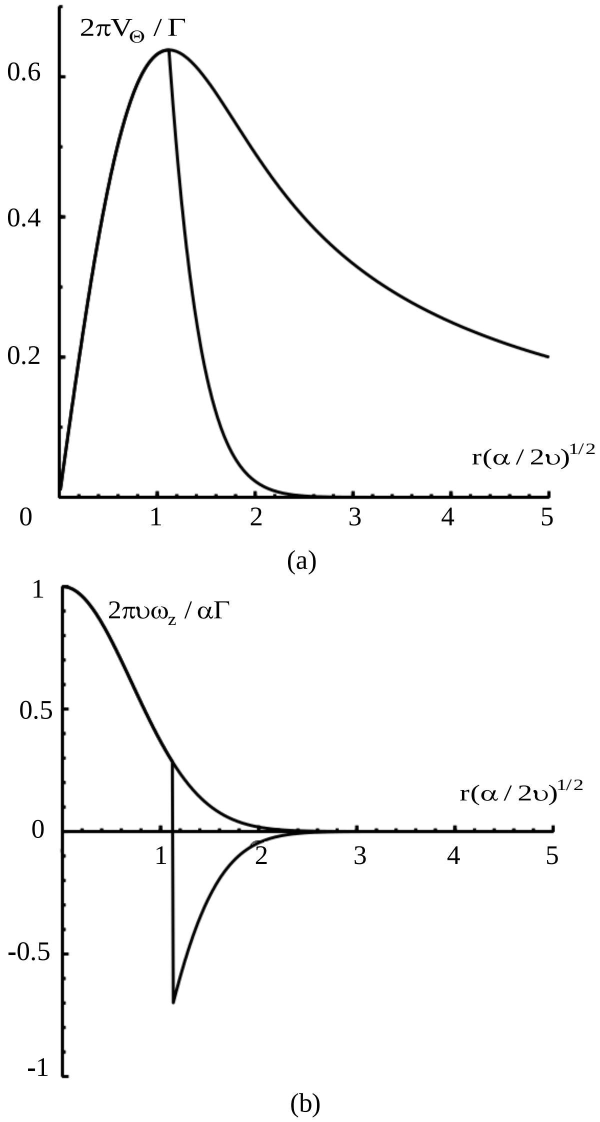

Fig. 1. Distributions of azimuthal velocity (a) and vorticity (b) in the Burgers-Rott vortex and its compact counterpart

The value r_k corresponds to the point of maximum for the function

$$V_\theta = \frac{\Gamma}{2\pi r}\left(1-\exp\left(-\frac{1}{2}\Lambda r^2\right)\right).$$

5. A compact Burgers vortex with a laminar core and a turbulent periphery

According to solution (29), the velocity field at the periphery of the Burgers-Rott vortex decreases faster than that of the point vortex (hyperbolic law). This, in turn, means the inertial instability of the incompressible fluid flow [48, 49]. It may not be by chance that Burgers considered the flow to be turbulent, although he actually used the Navier-Stokes equation of laminar motion (see equation in [35]). At that time (1948) direct numerical simulation of turbulence based on the Navier-Stokes equations was out of the question - computers simply did not exist yet. In order for the Burgers vortex model to correspond as closely as possible to reality, consider a combined vortex. The core of the vortex is a Burgers vortex (29), and the periphery is a turbulent flow based on relations (24) and (25), which now are understood as averaged velocity fields. The same as for the case of laminar flow, let's consider the general equations describing the turbulent type of flow. These are the Reynolds equations [50] in the cylindrical coordinate frame [40]:

$$\rho\left(\frac{\partial\bar V_r}{\partial t}+\bar V_r\frac{\partial\bar V_r}{\partial r}+\frac{\bar V_\theta}{r}\frac{\partial\bar V_r}{\partial\theta}+\bar V_z\frac{\partial\bar V_r}{\partial z}-\frac{\bar V_\theta}{r^2}\right)=$$
$$=-\frac{\partial\bar p}{\partial r}+\mu\left(\frac{\partial^2\bar V_r}{\partial r^2}+\frac{1}{r}\frac{\partial\bar V_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2\bar V_r}{\partial\theta^2}+\frac{\partial^2\bar V_r}{\partial z^2}-\right.$$
$$\left.-\frac{\bar V_r}{r^2}-\frac{2}{r^2}\frac{\partial\bar V_\theta}{\partial\theta}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(-r\rho\bar V_r'^2\right)+$$
$$+\frac{1}{r}\frac{\partial}{\partial\theta}\left(-r\rho\overline{V_r'V_\theta'}\right)+\frac{\partial}{\partial z}\left(-\rho\overline{V_r'V_z'}\right)-\frac{1}{r}\left(-\rho\bar V_\theta'^2\right);\qquad(35)$$

$$\rho\left(\frac{\partial\bar V_\theta}{\partial t}+\bar V_r\frac{\partial\bar V_\theta}{\partial r}+\frac{\bar V_\theta}{r}\frac{\partial\bar V_\theta}{\partial\theta}+\bar V_z\frac{\partial\bar V_\theta}{\partial z}-\frac{\bar V_r\bar V_\theta}{r^2}\right)=$$
$$=-\frac{\bar V_\theta}{r^2}+\frac{2}{r^2}\frac{\partial\bar V_r}{\partial\theta}\Bigg)+\frac{1}{r}\frac{\partial}{\partial r}\left(-\rho\overline{V_r'V_\theta'}\right)+$$
$$+\frac{1}{r}\frac{\partial}{\partial\theta}\left(-r\rho\bar V_\theta'^2\right)+\frac{\partial}{\partial z}\left(-\rho\overline{V_\theta'V_z'}\right)+\frac{2}{r}\left(-\rho\overline{V_r'V_\theta'}\right);\qquad(36)$$

$$\rho\left(\frac{\partial\bar V_z}{\partial t}+\bar V_r\frac{\partial\bar V_z}{\partial r}+\frac{\bar V_\theta}{r}\frac{\partial\bar V_z}{\partial\theta}+\bar V_z\frac{\partial\bar V_z}{\partial z}\right)=-\frac{\partial\bar p}{\partial z}\qquad(37)$$

$$\frac{\partial}{\partial r}\left(r\bar V_r\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(r\bar V_\theta\right)+\frac{\partial}{\partial z}\left(r\bar V_z\right)=0.\qquad(38)$$

Substitution of

$$\begin{cases}\bar V_r=-\alpha\cdot r,\\ \bar V_z=2\cdot\alpha\cdot z,\\ \bar V_\theta=\bar V_\theta(r)\end{cases}$$

for equations (35) – (37) results in

$$\rho\left(\alpha^2 r-\frac{\bar V_\theta}{r^2}\right)=-\frac{\partial\bar p}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho\bar V_r'^2\right),\qquad(39)$$

$$\rho\left(-\alpha r\frac{\partial\bar V_\theta}{\partial r}-\frac{\alpha\bar V_\theta}{r}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(-\rho\overline{V_r'V_\theta'}\right)+$$
$$+\frac{2}{r}\left(-\rho\overline{V_r'V_\theta'}\right);\qquad(40)$$

$$\rho4\alpha^2 z=-\frac{\partial p}{\partial z}+$$
$$+\frac{1}{r}\frac{\partial}{\partial r}\left(-\rho\overline{V_r'V_z'}\right)+\frac{\partial}{\partial z}\left(-\rho\bar V_z'^2\right).\qquad(41)$$

We use the standard Boussinesq hypothesis for such flows [51] for the description of Reynolds turbulent stresses

$$\tau_{ij}=-A_{ij}\frac{\partial\bar V_i}{\partial x_j}.\qquad(42)$$

In formular (42) A_{ij} are coefficients of turbulent diffusion.

$$\rho\left(-\alpha r\frac{\partial\bar V_\theta}{\partial r}-\frac{\alpha\bar V_\theta}{r}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(A_\theta\frac{d\bar V_\theta}{dr}\right)+$$
$$+\frac{2}{r}\left(A_\theta\frac{d\bar V_\theta}{dr}\right).\qquad(43)$$

Since in the work density is considered constant, the physical quantity has the following dimensions:

$$\left[\frac{A_\theta}{\rho}\right]=\frac{м^2}{c}\Rightarrow\left[\frac{A_\theta}{\rho}\right]=\mathrm{Const}\frac{r^2}{t}.\qquad(44)$$

We use (44) for the stationary problem and put:

$$A_\theta=A_0 r^2.\qquad(45)$$

According to (45), as the scale increases, the coefficient of turbulent viscosity increases in proportion to the square of the distance to the axis of rotation of the vortex. Despite the hypothetical character of relation (45), it should be noted that Burgers' vortex, which he himself considered a turbulent flow, was obtained under the assumption of constancy of the viscosity coefficient (see [35]). Substituting (45) into (43), we obtain

$$\frac{d^2\bar V_\theta}{dr^2}+\left(4+\Lambda_T\right)\frac{1}{r}\frac{d\bar V_\theta}{dr}+\Lambda_T\frac{1}{r^2}\bar V_\theta=0.\qquad(46)$$

$$\Lambda_T=\frac{\alpha\rho}{A_0}.$$

The general solution of equation (46) has the following is:

$$\bar{V}_\theta = C_1 r^{-\left(\frac{3}{2}+\frac{1}{2}\Lambda_T\right)+\frac{1}{2}\sqrt{9+2\Lambda_T+\Lambda_T^2}} + C_2 r^{-\left(\frac{3}{2}+\frac{1}{2}\Lambda_T\right)-\frac{1}{2}\sqrt{9+2\Lambda_T+\Lambda_T^2}}. \qquad (47)$$

By varying the parameter Λ_T, it is possible to obtain different velocity distributions. We will give the explicit form of the solution for the value

$$\Lambda_T = 1. \qquad (48)$$

Combining (47) with (48), we get:

$$\bar{V}_\theta = C_1 r^{-2+\sqrt{3}} + C_2 r^{-2-\sqrt{3}}. \qquad (49)$$

We determine the constants in (47) under the condition of "stitching" solutions (29) and (49) at $r = r_k$ as well as the condition of compactness of the velocity field (which means compensability of the vorticity field):

$$\bar{V}_\theta\left(R_V\right) = 0. \qquad (50)$$

According to what was said about the boundary conditions, the following solution is obtained:

$$\bar{V}_\theta = \frac{\Gamma}{2\pi r}\left(\frac{r}{r_k}\right)^{-1-\sqrt{3}} \frac{\left[1-\exp\left(-\frac{1}{2}r_k^2\right)\right]}{\left(1-\left(\frac{r_k}{R_V}\right)^{2\sqrt{3}}\right)} \times \left(1-\left(\frac{r}{R_V}\right)^{2\sqrt{3}}\right). \qquad (51)$$

The vorticity, according to (51), has the following form

$$\bar{\omega}_z = \frac{\left[1-\exp\left(-\frac{1}{2}r_k^2\right)\right]}{\left(1-\left(\frac{r_k}{R_V}\right)^{2\sqrt{3}}\right)} \frac{\Gamma}{2\pi r^{3+\sqrt{3}}} \times \left(-\left(1+\sqrt{3}\right)+\left(1-\sqrt{3}\right)\left(\frac{r}{R_V}\right)^{2\sqrt{3}}\right). \qquad (52)$$

In Fig. 2 the resulting distributions are presented. It will be recalled that piecewise continuous solutions have been used in hydromechanics for a long time [37]. And there is nothing seditious about it. Moreover, by specifying a vortex of finite dimensions at the initial moment, we can further monitor its evolution, and the system of equations will do its work by itself: all types of motions (modes) that are not characteristic of it will dissipate and leave only inherent ones (i.e. coherent) modes. It is important that the initial distribution of the vortex does not reach infinity, which grossly violates the law of conservation of energy: for a finite period of time, finite power is unable to create (transfer into motion) infinite energy. This shortcoming is characteristic of almost all existing models of the vortex flows.

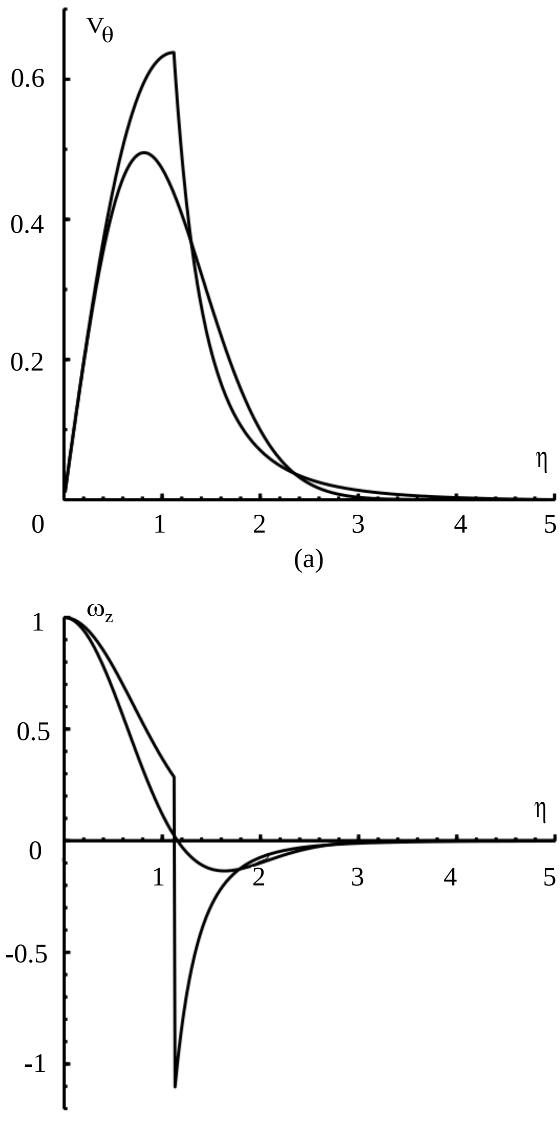

Fig. 2. Distributions of velocity (a) and vorticity (b) in the automodel solution (56), (57) and (51), (52)

Finally, let us note an interesting paper [52], where it is shown in particular that a Kármán vortex street cannot form in boundary layer, in its usual understanding.

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The background shear flow of the boundary layer enhances the formation of a vortex sheet of the same sign and makes the development of a vortex sheet with an opposite rotation impossible. That is why in Prandtl's report [8] observed formation vortex sheet of one sign in the boundary layers on the wings.

6. Von Kármán vortex street simulation based on models of compact vortex flows

Flow around a streamlined body due to instability generates the so-called Karman vortex street. An analytical expression for the complex potential can be found in textbooks on hydromechanics [2]

$$W=\frac{i\Gamma}{2\pi}\left(\ln\left(\sin\frac{\pi z}{a}\right)-\ln\left(\sin\frac{\pi}{a}\left(z-\frac{1}{2}a-ih\right)\right)\right), \qquad (53)$$

where Γ is a circulation; a,h are spatial scales.

The point vortex model used in (53), however, does not correspond to the physics of the phenomenon. Having detached from the body, vortices of the vortex street are formed only one by one, not in pairs. Moreover, as stated in [2] and shown above in this work, the kinetic energy of a point vortex is infinite. This statement is also true for the periphery of the Rankin vortex, the Burgers-Rott vortex, Ozeen, Sullivan and many others [53], since all the mentioned vortices have a hyperbolic distribution for velocity as an asymptote when they are far from the axis of rotation. Therefore, all the models just mentioned, in particular the point vortex and the Rankine vortex, do not correspond to the finite kinetic energy of each vortex in the vortex street, let alone compact nature of the vortices [38].

As an alternative to the point vortex and the Rankine vortex, the authors [38] used an isolated Gaussian to approximate Kármán vortex street [54]

$$V_\theta=V_0\cdot r\cdot\exp(-r^2). \qquad (54)$$

The vorticity field corresponding to (54) has domains of vorticity of different signs, like (34), (52). Indeed,

$$\omega_z=\frac{1}{r}\frac{d\left(rV_\theta\right)}{dr}=2(1-r^2)\exp\left(-r^2\right). \qquad (55)$$

Vorticity at the point $r=1$ (see (55)) changes the sign - from positive to negative. The authors of [38] consider model (54) to be significantly better than the point vortex model. Their argument is based on the fact that it is simply necessary to consider only two pairs of isolated Gaussian vortices instead of twenty or so pairs of point vortices.

Model (54) has, however, shortcomings. The size of the vortex core and the size of the vortex itself (in which the velocity drops to 1 ... 2 % compared to the maximum value) are strictly correlated by the ratio (54). Such a correlation also corresponds to the model based on Richardson's 4/3 law [55] turbulent vortex diffusion [28]

$$V_\theta=V_0 r\cdot\exp(-0.75r^2). \qquad (56)$$

The vorticity, according to (56), is

$$\omega_z=V_0\left(1-0.75r^2\right)\cdot\exp(-0.75r^2). \qquad (57)$$

The above mensioned drawback was overcome in the models of quasi-point laminar [29] and quasi-point turbulent [30] vortices. These models are based on solutions of the corresponding stationary Navier-Stokes equations in the form of Gromeka-Lamb. For a laminar flow, we have solution of equation (7) in the form of a quasi-point vortex [29]

$$V_\theta=\frac{\Gamma}{2\pi r}\left(1-\left(\frac{r}{R_V}\right)^2\right). \qquad (58)$$

For a turbulent flow, neglecting molecular diffusion, the Reynolds-averaged Navier-Stokes equations in the Griomeka-Lamb form lead to the following equation (partial case of (43) when there is only azimuthal velocity)

$$0=K_T\left(\frac{d^2\bar{V}_\theta}{dr^2}+\frac{2}{r}\frac{d\bar{V}_\theta}{dr}\right). \qquad (59)$$

The solution of equation (59), which is very close to a point vortex, has the form (details in [30])

$$\bar{V}_\theta=\frac{\Gamma}{2\pi r}\left(1-\frac{r}{R_V}\right). \qquad (60)$$

Both solutions, (58) and (60) can be used as initial distributions of the velocity field to simulate Karman vortex street. The analogue of formula (53) in the case of laminar flow has the form

$$V_\theta=\sum_{m=-\infty}^{\infty}\frac{-\Gamma}{2\pi\left[\left(x-ma\right)^2+y^2\right]^{1/2}}\left(1-\frac{\left(x-ma\right)^2+y^2}{R_V^2}\right)+$$

$$+\sum_{m=-\infty}^{\infty}\frac{-\Gamma}{2\pi\left[\left(x-(m+1/2)a\right)^2+(y-h)^2\right]^{1/2}}\times$$
$$\times\left(1-\frac{\left(x-(m+1/2)a\right)^2+(y-h)^2}{R_V^2}\right).$$

For the ultimate turbulent flow (Reynolds number is equal to several million and higher), the velocity field of the vortex street can be constructed based on the quasi-point model of the turbulent vortex [30]

$$V_\theta=\sum_{m=-\infty}^{\infty}\frac{-\Gamma}{2\pi\left[(x-ma)^2+y^2\right]^{1/2}}\left(1-\frac{(x-ma)^2+y^2}{R_V^2}\right)+$$
$$+\sum_{m=-\infty}^{\infty}\frac{-\Gamma}{2\pi\left[\left(x-(m+1/2)a\right)^2+(y-h)^2\right]^{1/2}}\times$$
$$\times\left(1-\frac{\left[\left(x-(m+1/2)a\right)^2+(y-h)^2\right]^{1/2}}{R_V}\right). \quad (61)$$

To describe the non-stationary Kármán vortex street, one can use the solution of the problem of the generation of a turbulent vortex by a thin cylinder [31] in the approximation of the constancy of the turbulent diffusion coefficient. Obtained in [31] asymptotic solution for the following unsteady Navier-Stokes equation in Gromeka-Lamb form, averaged over Reynolds

$$\frac{\partial\bar{V}_\theta}{\partial t}=\frac{1}{Re}\left(\frac{\partial^2\bar{V}_\theta}{\partial r^2}+\frac{2}{r}\frac{\partial\bar{V}_\theta}{\partial r}\right). \quad (62)$$

Solution (62) has the form:

$$\bar{V}_\theta(r,t)=\frac{\Gamma}{2\pi r}\left[1-\mathrm{erf}\left(\sqrt{\frac{Re}{t}}\frac{(r-R_i)}{2}\right)\right]. \quad (63)$$

Here R_i is a radius of the cylinder generating vortex.

Equation (62) and its solution (63) in dimensionless quantities correspond to the problem of the asymptotic (for sufficiently large moments of time) behavior of a turbulent vortex generated by a cylinder of small radius R_i. This solution is graphically very close to (60) and to the distribution in a point vortex. At the same time, it is not stable and compact according to Saffman. Applying (63) to the description of the Karmán vortex street, we can obtain

$$\sum_{m=-\infty}^{\infty}\frac{-\Gamma}{2\pi\left[(x-ma)^2+y^2\right]^{1/2}}\times$$
$$\times\left(1-\mathrm{erf}\left(\sqrt{\frac{Re}{t}}\frac{\left[(x-ma)^2+y^2\right]^{1/2}}{2}\right)\right)+$$
$$+\sum_{m=-\infty}^{\infty}\frac{-\Gamma}{2\pi\left[\left(x-(m+1/2)a\right)^2+(y-h)^2\right]^{1/2}}\times$$
$$\times\left(1-\mathrm{erf}\left(\sqrt{\frac{Re}{t}}\frac{\left[\left(x-(m+1/2)a\right)^2+(y-h)^2\right]^{1/2}}{2}\right)\right).$$

Among modern works, the work [56] deserves mention, in which the author proposes a robust model of the Kármán vortex street. The model has no singularities on the axis of the vortex. There is also another model for persistent Kármán vortex steet. It uses a generalized model of a compact compensated turbulent vortex - solutions of equation (61) [32]. The velocity field can be represented as

$$V_\theta=-\sum_{m=-\infty}^{\infty}V_\theta^{Ut}\left((x-ma)^2+y^2\right)+$$
$$+\sum_{m=-\infty}^{\infty}V_\theta^{Ut}\left(\left(x-(ma+1/2)\right)^2+y^2\right),$$

where $V_\theta^{Ut}=\begin{cases}\dfrac{\Gamma}{2\pi r}, & 0\le r\le\varepsilon;\\ \dfrac{\Gamma}{2\pi(R_V-\varepsilon)}\left(\dfrac{R_V}{r}-1\right), & \varepsilon<r\le R_V;\\ 0, & r>R_V.\end{cases}$

Discussion

Despite the long-standing (starting with the famous work of Helholtz in 1858) history of vortex dynamics, there are certain significant shortcomings in this field of science. They are related to the fact that a phenomenon with a more complex physical essence (nature) is replaced (unreasonably) by much simpler ones. In particular, we are talking about replacing a significantly viscous vortex motion with a non - viscous one. At the same time, the mathematical simplicity of the description of inviscid motion is implicitly understood. But, as shown in this work, this simplification is groundless and has negative consequences. Being based on the model of inviscid fluid flow, it is impossible to physically interpret

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the formation of a vortex sheet in the boundary layer, as well as its further existence in the form of free compact vortices. The generation of the vortex sheet and the late stage of its existence (free vortex) are of great importance for numerical modeling [3]. They, in fact, play the role of the initial and conditionally final boundary conditions in time and make it possible to test various complex models of the formation and development of vortex flows that take place when a flow of a viscous liquid flows around wing and body as a whole.

A significant step in understanding the physics of the boundary layer was the discovery of the fact that molecular viscosity is a variable value for the motion of a body in a fluid. This fact certainly opens up new possibilities in the study of the boundary layer and the flows formed in it.

Conclusions

When fluid flows around a wing or a body of finite thickness, typical vortex structures are formed: a vortex sheet (in the viscous boundary layer), vortices on the leading edge of the wing (or body [57]), as well as the Kármán vortex street. Confusion or simply misunderstanding in the field of vortex dynamics, when viscous vortex motion is considered non-viscous, led to the emergence of scientific approaches (method of discrete vortices, etc.), where the physical nature of vortex formation is not taken into account. The velocity field is approximated by artificially superimposing a pair of vortex flows (dipole), instead, as stated in this paper, the boundary layer strengthens vortices of one direction of rotation and disables (suppresses) the opposite one. In addition, it is also noted, with reference to the sources, that during the formation of vortices in the boundary layer at the stage of their detachment from the surface of the body, the process of formation of a domain of vorticity of reversed sign occurs, which enables the existence of compact (isolated) vortex flows. The process of vortex sheet formation is associated with the Kelvin-Helmholtz instability and the inhomogeneity of the velocity field along the surface of the body (the region of the flow development). None of the known models used to simulate leading edge vortices or the vortex street, as well as free vortices that have left the body, including the Kármán vortex street, are currently not modeled by compact vortices - those in which the velocity field is concentrated in a finite domain. Often they are also called isolated vortices. The nature of the formation of geophysical vortices (atmosphere and ocean) has led to the understanding that real vortices cannot have infinite kinetic energy, and therefore the existing models of vortices should be improved, making them such that they do not contradict the fundamental law of nature - the conservation and transformation of energy. The first attempt to use a compact analogue of the Renkin vortex was made, without any resoning, by Stern [58]. But his work remained without due attention from experts in vortex dynamics. Independently of Stern, fully substantiated, from a mathematical and physical point of view, the cited models of compact vortices were developed by one of the authors of the paper during approximately 15 previous years. This experience helped to understand the well-known model of the Burgers-Rott vortex, which is widely used in aviation problems. Since the Burgers-Rott vortex, as well as the Rankine vortex, is not compact, it was proposed solely on the basis of the general solution obtained by Burgers to make the flow compact, i.e., one that exists within a finite domain. Although such a vortex is composed, nevertheless momentum conservation equation (Navier-Stokes in the Gromeka-Lamb form) is valid at every point in flow domain. It is ponted out, with reference to reputable scientific sources, that such an approach has long been used in fluid and gas mechanics - a class of functions in which not only the velocity field, but also its derivatives are continuous does not allow simulating flows quite simply and solving problems accordingly.

Another essential point is that the peripheries of vortex flows, such as the Burgers-Rott vortex and others, are unstable. Even the title of Burgers' paper [35] refers to a turbulent flow. Therefore, instead of the periphery of the Burgers-Rott vortex, which corresponds to a laminar flow, it was proposed a more realistic model – a vortex with a laminar core and a turbulent periphery. At the same time, the dimensionality of the coefficient of turbulent diffusion was used and the assumption was made that in a stationary turbulent flow turbulent diffusion is proportional to the square of the distance to the axis of rotation - as in the self-similar variable used in the theory of vortex diffusion. This assumption made it possible to build a model in which the velocity and vorticity fields are quite close to those obtained by the model in which Richardson's law is used for turbulent diffusion in a stratified medium (atmosphere, ocean).

As for the vortex sheet itself, the possibility of a turbulent type of motion in the form of vortices rolling along the outer (almost stationary) boundary of the boundary layer was shown, which correlates with the phenomenon of Kelvin-Helmholtz instability and, most importantly, enables the existence of a system of vortices with one direction of rotation - without adding fictitious one as in the method of discrete vortices.

Finally, with regard to the Kármán vortex street, different models of compact vortex flows - both laminar and turbulent - have also been proposed to describe it.

As further research, it is possible to consider the application of the proposed models of compact vortex flows in numerical simulations and compare them with the results of other studies.

Contribution of authors: conceptualization – **Pavlo Lukianov**, **Lin Song**; formulation of task – **PavloLukianov, Lin Song**; analysis – **Pavlo Lukianov**, **Lin Song**; software – **Lin Song**; development of mathematical models – **Pavlo Lukianov**; analysis of results – **Pavlo Lukianov**, **Lin Song**.

All the authors have read and agreed to the published version of the manuscript.

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КОМПАКТНІ АНАЛОГИ МОДЕЛЕЙ ВИХРОВИХ ТЕЧІЙ, ЩО ВИНИКАЮТЬ ПІД ЧАС ПОЛЬОТУ ЛІТАКА

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Предметом даної роботи є розробка компактних аналогів моделей вихрових течій, які використовуються при моделюванні вихрових утворень, що спостерігаються під час польоту літального апарату та руху тіла в рідині. Зокрема, виділено два істотних непорозуміння, що панують у цій галузі науки. Перше непорозуміння полягає в тому, що стаціонарний рух частинок рідини по колу розглядається як нев'язкий вихор. Таким чином, будь-яка модель вихрової течії, яка явно не містить в'язкість, вважається такою, що описує нев'язкий вихровий рух. Доведено, що це не так: стаціонарному в'язкому руху частинок рідини по кругових орбітах відповідає самоврівноваження однієї сили - сили в'язкості. Такий висновок у явній формі зроблено вперше. І це дуже важливо, оскільки змінює наші уявлення про баланс сил, де неодмінно повинні бути присутні дві або більше сили різної природи. Саме подолання цього непорозуміння відкриває шлях до створення компактних аналогів існуючих моделей вихрових рухів. Попутно було усунено ще одне - друге загальне непорозуміння в області динаміки вихорів. Де б ми не читали, ми побачимо, що компактність вихрового потоку ототожнюється з компактністю поля завихреності. Цьому сприяє те, що розглядаються рівняння для завихреності, а не для швидкості. В результаті, за винятком однієї або двох, всі моделі вихорів відповідають обертанню всього простору, аж до нескінченності, порушуючи фундаментальний закон фізики - закон збереження і перетворення енергії. Йдеться про те, що в якості другого непорозуміння допускається помилка при розрахунку кінетичної енергії вихрового струму: не враховується якобіан у циліндричних (полярних) координатах. В результаті всі згадані моделі вихрових течій, що відповідають гіперболічному закону як їх асимптотика на периферії, мають нескінченну кінетичну енергію. Звичайно, це не відповідає утворенню та еволюції компактних вихрових структур. Тому в роботі, на основі подолання зазначених непорозумінь, представлено низку як раніше отриманих моделей компактних вихрових струмів, так і вперше отриманих. Зокрема, це стосується турбулентної вихрової течії при формуванні вихрового шару, компактних аналогів вихору Бюргерса-Ротта - як класичного, що відповідає ламінарному руху, так і такого, що складається з ламінарного потоку в ядрі та турбулентного потоку. на периферії вихору. **Методи дослідження** суто теоретичні. Використовуються відомі теореми теоретичної механіки, математичної теорії поля, варіаційного числення тощо. Отримані розв'язки порівнюються з існуючими відповідними аналогами некомпактних течій. **Висновки.** Використовуючи методи варіаційного числення, вдалося показати можливість формування обертального руху квазітвердого тіла в прикордонному шарі нестисливої рідини. Сама наявність в'язкості, а точніше її врахування (примежовий шар), свідчить про можливий перехід течії від плоскопаралельного руху до щойно згаданого обертального через нестійкість Кельвіна-Гельмгольца. Крім того, в роботі отримано дві нові моделі вихрової течії Бюргерса-Ротта. У першій моделі використовується загальний розв'язок, отриманий Бюргерсом, але ця модель відповідає комбінованому вихору: хоча поле швидкостей в ній безперервне, поле завихреності має розрив - в точці максимуму поля швидкостей. Доведено, що такий підхід цілком можливий: рівняння руху виконується всюди, тобто в кожній точці простору дотичні напруження є неперервними функціями. Оскільки периферія вихору Бюргерса-Ротта є нестійкою течією, пропонується інша модель – з ламінарним ядром і турбулентною периферією. Звичайно, рух частинок рідини в периферійній області описується розподілом швидкостей, відмінним від розподілу швидкості Бюргерса. Нарешті, розглянуто можливе використання відомих моделей компактних вихрових течій при моделюванні вихрової доріжки фон Кармана. Із зазначенням переваг цих моделей.

Ключові слова: літак; вихрові течії; вихор Бюргерса-Ротта; вихрова пелена; вихрова доріжка Кармана; два нерозуміння в вихровій динаміці.

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